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STRENGTH ANALYSIS OF COMPOSITE PLATES

IN A THERMAL ENVIRONMENT

by

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An investigation of the strength of a fiber reinforced composite plate subjected to external loads and thermal environment is considered. The analysis is restricted to symmetric layups. The general Tsai-Wu failure criterion is used to predict failure. The analyses has resulted in a computer program which will predict the loads at which a composite plate will fail under a specified thermal environment.

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# NOTATION

$a,b,c$..........Eq.(64) coefficients, defined by Eq's.(65)-(67)

$\underset{\approx}{A}$..............laminate inplane stiffness matrix; Eq.(34)

$\underset{\approx}{B}$..............laminate coupling stiffness matrix; Eq.(35)

$\underset{\approx}{C}_i$.............lamina stiffness matrix; Eq.(1)

$\underset{\approx}{D}$..............laminate moment stiffness matrix; Eq.(36)

$\underset{\sim}{e}$..............strain

$E_x,E_y$..........lamina longitudinal and transverse stiffnesses

$F_{xx},F_x,F_{yy},F_y,F_{xy},F_{ss}$...Tsai-Wu failure parameters; Eq's.(56) and (58)

$G_{xy}$............lamina shear modulus

$\underset{\sim}{\bar{M}}$..............laminate moment vector; Eq.(26)

$\underset{\sim}{\bar{M}}_T$.............laminate thermal moment vector; Eq.38)

$\underset{\sim}{\bar{N}}$..............laminate force vector; Eq.(25)

$\underset{\sim}{\bar{N}}_T$.............laminate thermal force vector; Eq.(37)

$\underset{\approx}{Q}_i$.............lamina stiffness matrix; Eq.(13)

$R_i$.............lamina strength ratio; Eq.(60)

$\underset{\approx}{S}_i$.............lamina compliance matrix; Eq.(13)

$t$..............plate thickness

$T_i$.............lamina temperature

$u,v,w$..........displacements along the x,y,z axes

$\underline{x}_i,\underline{y}_i,z_i$.......lamina axes

$\bar{x},\bar{y},\bar{z}$..........laminate axes

$\underset{\sim}{\alpha}$..............vector of thermal coefficients of expansion

$\underset{\sim}{\beta}$..............vector of thermal coefficients

$\gamma$..............engineering shear strain

$\underset{\sim}{\kappa}$..............laminate curvature vector

$\phi_i$.............orientation angle of i-th ply

$\theta_i$.............temperature of i-th ply; Eq.(2)

$\nu_{xy},\nu_{yx}$........longitudinal and transverse Poisson's ratio

$X,X',Y,Y',S$....lamina strengths

LIST OF FIGURES AND TABLES

## 1.0 Introduction

In a severe thermal environment, materials generally experience a degraduation of stiffness and strength, and as a result there is a diminishment in the load carrying capacity of the material. In order to design a structure which will perform safely in a thermal environment, the structural engineer must have an analysis program which will predict the behavior of the structure. This report documents an analysis for the behavior of a fiber reinforced composite plate subjected to loads in a thermal environment.

## 2.0 Description of Fiber Reinforced Composite Plates

As a result of imperfections, such as dislocations and surface scratches, the strength of bulk material is orders of magnitude less than the strength of the material in crystal or whisker form. The cost of producing crystals is so great as to prohibit production except on a limited basis. On the other hand, fibers of a material can be produced at a cost which permits their application to structural systems.

Depending upon it's diameter, a fiber of a material has a strength intermediate to the bulk and crystal strengths of the material. For example, the strength of steel fiber is approximately 4.14 MPa (600,000 psi), compared to a strength of 1.04 MPa (150,000 psi) for steel in it's common bulk form. Thus the use of steel fiber in place of common steel results in a four-fold reduction in weight. Actually the use of fiber requires their embedment in a matrix of another material in order to achieve an useful structural form. However even with embedment, fiber reinforced materials offer a significant increase in the strength to weight ratio over common materials. This is the primary reason for using composite materials.

One of the most common useful structural component is the plate element. A fiber reinforced composite plate (or simply, a laminate) consists of layers of lamina (or plies)

stacked one upon another. A lamina itself is comprised of unidirectional fibers embedded in a binding material, called the matrix. Generally, the binding matrix material is a much weaker and more ductile material than the fiber reinforcement embedded within it. Some common fiber/matrix combinations are graphite/epoxy, glass/epoxy, and boron/aluminum. Some idea of the dimensions involved can be obtained from the dimensions of a typical graphite/epoxy lamina;

i) fiber diameter is approximately .008 mm. (.0003 in.)

ii) lamina thickness is approximately .125 mm. (.005 in.)

iii) fiber volume to total volume (called the fiber fraction) is about 0.5

For these dimensions, a 12.5 mm. (0.5 in.) thick laminate will consist of 100 plies (lamina) stacked one upon another.

As a result of unidirectional fibers embedded within it, a lamina is not isotropic. The stiffness of a lamina along it's fiber direction (called the longitudinal stiffness) is much greater than the stiffness of the lamina in the direction orthogonal to the fiber (called the transverse stiffness). For example, in the case of a typical graphite/epoxy lamina, the longitudinal stiffness ($E_x$ or $E_L$) and the transverse stiffness ($E_y$ or $E_T$) are 181 GPa ($26 \times 10^6$ psi) and 10.3 GPa ($1.5 \times 10^6$ psi), respectively.

Stacking lamina of different orientation and thickness upon one another to form a composite plate results in an orthotropic body. One of the most attractive features of a composite plate is the capacity for design of material properties, an option not previously available to structural engineers.

## 3.0 Lamina and Laminate Coordinate Systems

In dealing with fiber reinforced composite plates, it is necessary to employ two types of coordinate systems. The first type of coordinate system is associated with a lamina. This coordinate system is referred to as the natural coordinate

system, or the local coordinate system, or the lamina (or ply) coordinate system. Here we assign the cartesian pair of x,y axes to the plane of the lamina, with the x-axis along the fiber direction of the lamina. Since adjacent lamina have different fiber orientation with respect to one another, there are as many natural coordinate systems as there are plies. The i-th lamina (ply) has the coordinate system $(x,y)_i$ or $(x_i,y_i)$. This is shown in Fig.1 below.

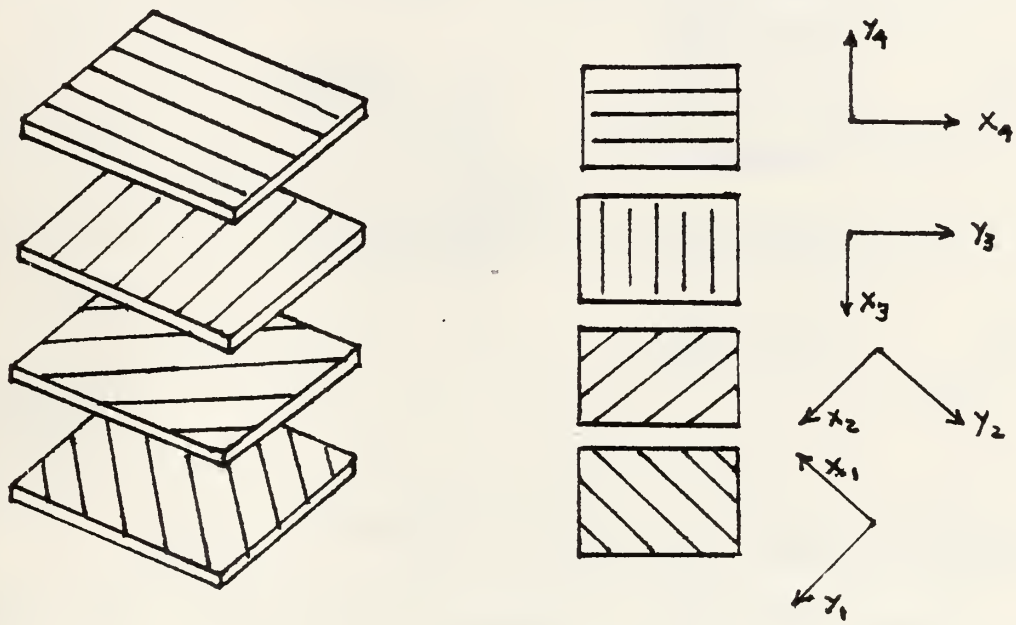



Figure 1. Lamina Layup to Form a Laminate

The second type of coordinate system associated with a composite plate is the coordinate system which ties all of the lamina coordinate systems to a single pair of axes, called the laminate or global coordinate system, and denoted by $(\bar{x},\bar{y})$. These axes lie in a plane parallel to the lamina planes, and are generally located at the midplane of the laminate, as shown in Fig.2 for a 3 ply laminate.

Each of the local (lamina) coordinate systems are related to the global (laminate) coordinate system by the angle $\phi_i$, where $\phi_i$ is the angle between the $x_i$ axis of the i-th

lamina and the $\bar{x}$ axis of the laminate. In this way, the orientation layup of a n-ply laminate can be denoted by the notation $[\phi_1, \phi_2, \phi_3, \ldots, \phi_n]$. There is no requirement that

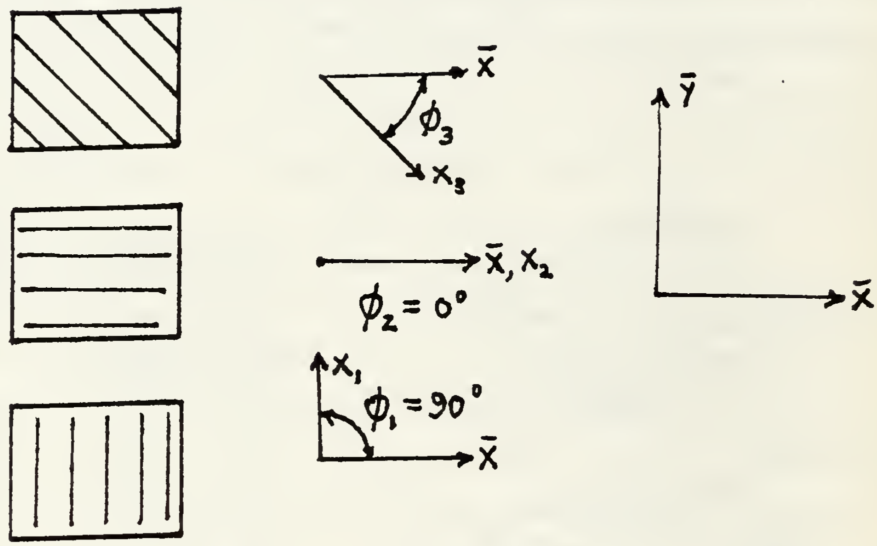



Figure 2. Lamina and Laminate Coordinate Systems

the lamina be of equal thickness, although this is a common construction.

## 4.0 Stress-Strain Relations in Natural Coordinates for a Lamina in Plane Stress

In the case of plates subjected to inplane and transverse loads, the result is the plane stress state. This means that for a lamina in the x,y plane, interlaminar stresses $\sigma_{zz}$, $\sigma_{zy}$, and $\sigma_{zx}$ are negligible compared to inplane stresses $\sigma_{xx}$, $\sigma_{yy}$, and $\sigma_{xy}$, and are therefore ignored. It should be noted in passing that interlaminar stresses can be very large in local regions of boundaries. (See Ref.1 for details). In this report interlaminar stresses are not considered.

In terms of it's own natural coordinate system, the stress-strain relations for the i-th lamina, in tensor notation, are

$$(\sigma_{jk})_i=(C_{jkrs}e_{rs}-\beta_{jk}\theta)_i \qquad j,k,r,s=x_i,y_i \tag{1}$$

where $\sigma_{jk}$ and $e_{rs}$ are the second order stress and strain tensors, $\beta_{jk}$ is the second order tensor of thermal coefficients, $C_{jkrs}$ is the fourth order tensor of stiffness coefficients, and $\theta$ is the difference in temperature from the stress free temperature. That is, if T is the stress free (curing) temperature, and $T_i$ is the temperature of the i-th lamina, then

$$\theta_i=(T_i-T) \tag{2}$$

Throughout this report, the Einstein summation convention is in effect for all tensor equations.

Alternatively, Eq.(1) may be written for strains in terms of stress, as

$$(e_{jk})_i=(S_{jkrs}\sigma_{rs}+\alpha_{jk}\theta)_i \qquad j,k,r,s=x_i,y_i \tag{3}$$

where $S_{jkrs}$ is the fourth order tensor of compliance coefficients, and $\alpha_{jk}$ is the tensor of thermal expansion coefficients. We note that properties of a lamina, such as stiffnesses, compliances, thermal coefficients, and thermal expansion coefficients, are functions of temperature. However over narrow ranges of temperature, properties can be treated as constant without introducing significant error into the analysis.

As a matter of convenience, hereafter we proceed using contracted "vector" notation. It should always be kept in mind that stresses, strains, thermal coefficients, thermal expansion coefficients, stiffnesses, and compliances are, in fact, tensors, not vectors, and therefore obey the appropriate tensor transformation laws when transformations occur. The second order tensors ($\sigma_{ij}$, $e_{ij}$, $\beta_{ij}$, and $\alpha_{ij}$) in the contracted notation have a single subscript in accordance with Table 1, and the fourth order tensors ($S_{ijkl}$ and $C_{ijkl}$) carry the double subscript notation shown in Table 2.

| Tensor Subscripts | Contracted Notation |
|---|---|
| xx | 1 |
| yy | 2 |
| xy and yx | 6 |

Table 1. Contracted Notation for $2^{nd}$ order Tensors

| Tensor Subscripts | Contracted Notation |
|---|---|
| xxxx | 11 |
| xxyy | 12 |
| xxxy and xxyx | 16 |
| yyxx | 21 |
| yyyy | 22 |
| yyxy and yyyx | 26 |
| xyxx and yxxx | 61 |
| xyyy and yxyy | 62 |
| xyxy and yxyx | 66 |

Table 2. Contracted Notation for $4^{th}$ order Tensors

For example, using the symmetry conditions, $e_{xy}=e_{yx}$, $C_{xxxy}= C_{xxyx}$ and so on, and defining the engineering shear strain $\gamma_{xy}=(e_{xy}+ e_{yx})$, equations (1) become

$$\begin{aligned} &[\sigma_1 = C_{11}e_1 + C_{12}e_2 + C_{16}\gamma_6 - \beta_1\theta]_i \\ &[\sigma_2 = C_{21}e_1 + C_{22}e_2 + C_{26}\gamma_6 - \beta_2\theta]_i \\ &[\sigma_6 = C_{61}e_1 + C_{62}e_2 + C_{66}\gamma_6 - \beta_6\theta]_i \end{aligned} \tag{4}$$

or, in vector notation

$$\underset{\sim}{\sigma}_i = ( \underset{\approx}{C}\underset{\sim}{e} - \underset{\sim}{\beta}\theta )_i \tag{5}$$

where the i subscript denotes the equation is for the i-th lamina. In equation (5), and hereafter, a single tilda under a quantity denotes the quantity is treated as a 3x1 vector, a double tilda under a quantity means the quantity is treated as a 3x3 matrix, and a quantity without a tilda beneath it denotes that the quantity is a scalar.

As a result of the natural coordinates being along and orthogonal to the fibers of a lamina, the natural coordinates are the principal axes of the lamina. Thus the normal and shear effects uncouple, and Eq. (5) takes on the special form,

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{Bmatrix}_i = \begin{bmatrix} C_{11} & C_{12} & 0 \\ C_{21} & C_{22} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}_i \begin{Bmatrix} e_1 \\ e_2 \\ \gamma_6 \end{Bmatrix}_i - \theta_i \begin{Bmatrix} \beta_1 \\ \beta_2 \\ 0 \end{Bmatrix}_i \tag{6}$$

That is, in the natural coordinates of a lamina, $C_{16}=C_{26}=C_{66}=0$, and $\beta_6=0$. Furthermore, as the xy plane is a plane of symmetry, $C_{12}=C_{21}$, and therefore $\underset{\approx}{C}$ is a symmetric matrix. Thus a lamina has four independent stiffness coefficients ($C_{11}, C_{12}, C_{22}$, and $C_{66}$), and two independent thermal coefficients ($\beta_1$, and $\beta_2$). Such materials are called orthotropic materials. It is important to note that the special form of $\underset{\approx}{C}$ in Eq.(6), associated with uncoupled normal and shear behavior, is valid only in natural coordinates of a lamina.

The notation used most frequently in the composites literature is $\underset{\approx}{S}$ for compliance, and $\underset{\approx}{Q}$ rather than $\underset{\approx}{C}$ for stiffness. Hereafter we shall use $\underset{\approx}{S}$ and $\underset{\approx}{Q}$. In contracted notation, Eq's.(1) and (3) are,

$$\underset{\sim}{\sigma}_i = (\underset{\approx}{Q}\underset{\sim}{e} - \underset{\sim}{\beta}\theta)_i \tag{7}$$

and

$$\underset{\sim}{e}_i = (\underset{\approx}{S}\underset{\sim}{\sigma} + \underset{\sim}{\alpha}\theta)_i \tag{8}$$

Premultiplication of Eq.(7) by $\underset{\approx}{Q}^{-1}$ and solving for strain $\underset{\sim}{e}$ gives

$$\underset{\sim}{e}_i = (\underset{\approx}{Q}^{-1}\underset{\sim}{\sigma} + \underset{\approx}{Q}^{-1}\underset{\sim}{\beta}\theta)_i \tag{9}$$

A comparison of equations (8) and (9) shows the relations between $\underset{\approx}{Q}$ and $\underset{\approx}{S}$, and $\underset{\sim}{\beta}$ and $\underset{\sim}{\alpha}$ are

$$\underset{\approx}{Q} = \underset{\approx}{S}^{-1} \tag{10}$$

and

$$\underset{\sim}{\alpha} = \underset{\approx}{Q}^{-1}\underset{\sim}{\beta} \ . \tag{11}$$

The stiffness coefficients $Q_{jk}$ are related to the common engineering coefficients $E_x, E_y, \nu_{xy}$, and $G_{xy}$, by Eq's.(12). The engineering coefficients are obtained by performing the simple tension and shear tests on a lamina as depicted in

Figure 3. Pulling a lamina along the fiber direction (Fig.3a) gives the longitudinal stiffness, $E_x$. The tension test in the orthogonal direction (Fig.3b) gives the transverse stiffness $E_y$. The longitudinal Poisson's ratio $\nu_{xy}$ is the ratio of $e_x$ to $e_y$ obtained in the longitudinal tension test. The longitudinal shear modulus $G_{xy}$ is obtained from the shear test shown in Fig.3c. The relations between $S_{jk}$ and engineering coefficients are,

$$S_{11}=(1/E_x)$$
$$S_{12}=S_{21}=-(\nu_{xy}/E_x)=-(\nu_{yx}/Ey) \tag{12}$$
$$S_{22}=(1/E_y)$$
$$S_{66}=(1/G_{xy})$$

Note that the second of these relations defines a transverse Poisson's ratio, $\nu_{yx}$. The x and y subscripts denoting longitudinal and transverse properties are quite often replaced by T and L subscripts, i.e., $E_L$ denotes the longitudinal stiffness of a lamina, $E_T$ is the transverse stiffness, and so on. Using Eq.(10), the relations between stiffness coefficients $Q_{ij}$ and the engineering constants are found to be,

$$Q_{11}=(E_x/(1-\nu_{xy}\nu_{yx}))$$
$$Q_{12}=Q_{21}=(\nu_{xy}E_x/(1-\nu_{xy}\nu_{yx}))=(\nu_{yx}E_y/(1-\nu_{xy}\nu_{yx}))$$
$$Q_{22}=(E_y/(1-\nu_{xy}\nu_{yx}) \tag{13}$$
$$Q_{66}=G_{xy}$$

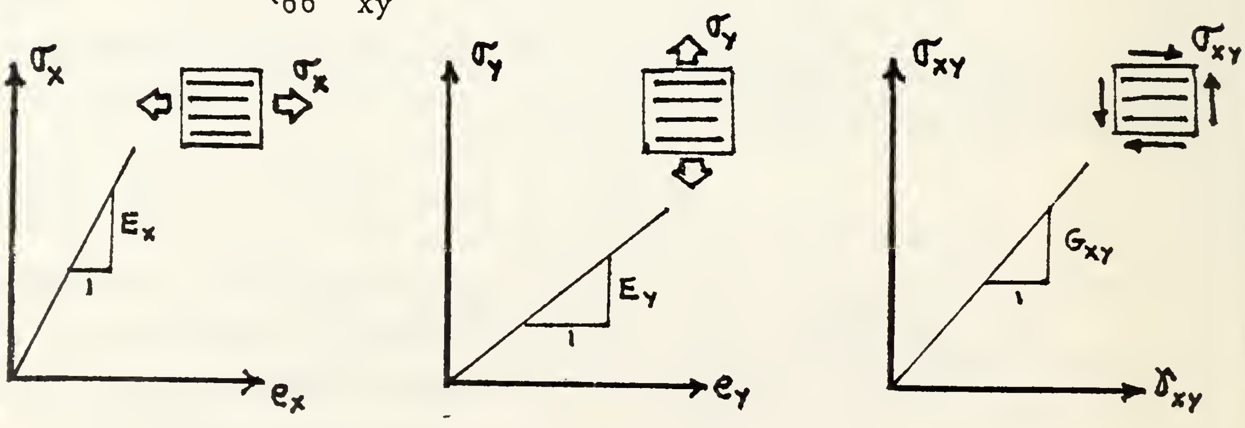



Figure 3. Experimental Tests for Engineering Stiffness Constants

## 5.0 Stress-Strain Relations for a Lamina in Plane Stress with Respect to Laminate Axes

In order to develope the equations for a laminate, it is necessary to relate each lamina comprising the laminate to a common coordinate system, the laminate axes. This is accomplished by coordinate transformation from lamina axes to laminate axes. Each lamina coordinate system $(x,y,z)_i$ is oriented to the laminate axes $(\bar{x},\bar{y},\bar{z})$, where $z_i$ and $\bar{z}$ coincide, by an angle $\phi_i$. Recall from Fig.2, that $\phi_i$ is the angle between the $x_i$ axis of the i-th lamina and the $\bar{x}$ axis of the laminate.

Hereafter all quantities with a bar over the quantity symbol means that the quantity is with respect to laminate axes, and quantities without a bar over them means the quantities are with respect to lamina axes. Thus the stresses, strains, thermal coefficients, and stiffnesses with respect to the laminate coordinate system are $\bar{\underset{\sim}{\sigma}}, \bar{\underset{\sim}{e}}, \bar{\underset{\sim}{\beta}}$, and $\bar{\underset{\approx}{Q}}$. Using the tensor transformation laws for the second order tensors,

$$
\begin{aligned}
(\sigma_{jk})_i &= (l_{jm} l_{kn} \bar{\sigma}_{mn})_i \\
(e_{jk})_i &= (l_{jm} l_{kn} \bar{e}_{mn})_i \qquad (14) \\
(\beta_{jk})_i &= (l_{jm} l_{kn} \bar{\beta}_{mn})_i
\end{aligned}
$$

where $l_{rs}$ is the cosine of the angle between the r-th lamina axis and the s-th laminate axis, gives the expressions for lamina quantities with respect to laminate axes. Denoting the transformation between coordinate systems by the matrix $\underset{\approx}{T}$, Eq's.(14) may be written,

$$
\begin{aligned}
\underset{\sim}{\sigma}_i &= (\underset{\approx}{T}\bar{\underset{\sim}{\sigma}})_i \\
\underset{\sim}{e}_i &= (\underset{\approx}{T}\bar{\underset{\sim}{e}})_i \qquad (15) \\
\underset{\sim}{\beta}_i &= (\underset{\approx}{T}\bar{\underset{\sim}{\beta}})_i
\end{aligned}
$$

Substitution of Eq's.(15) into Eq's.(7) gives

$$\underset{\approx}{T}_i \underset{\sim}{\bar{\sigma}}_i = \underset{\approx}{Q}_i \underset{\approx}{T}_i \underset{\sim}{\bar{e}}_i - \underset{\approx}{T}_i \underset{\sim}{\bar{\beta}}_i \theta_i$$

Premultiplication through by the inverse of $\underset{\approx}{T}_i$ gives,

$$\underset{\sim}{\bar{\sigma}}_i = \underset{\approx}{T}_i^{-1} \underset{\approx}{Q}_i \underset{\approx}{T}_i \underset{\sim}{\bar{e}}_i - \underset{\sim}{\bar{\beta}}_i \theta_i$$

Denoting the matrix $\underset{\approx}{T}_i^{-1} \underset{\approx}{Q}_i \underset{\approx}{T}_i$ by $\underset{\approx}{\bar{Q}}_i$, where $\underset{\approx}{\bar{Q}}_i$ is the 3x3 stiffness matrix of the i-th lamina with respect to laminate axes, we obtain the stress-strain relations for a lamina in terms of laminate axes as,

$$\underset{\sim}{\bar{\sigma}}_i = \underset{\approx}{\bar{Q}}_i \underset{\sim}{\bar{e}}_i - \underset{\sim}{\bar{\beta}}_i \theta_i$$

Recalling Eq.(11), yields the more useful expression (since the coefficients of thermal expansion are generally known),

$$\underset{\sim}{\bar{\sigma}}_i = \underset{\approx}{\bar{Q}}_i (\underset{\sim}{\bar{e}}_i - \underset{\sim}{\bar{\alpha}}_i \theta_i) \tag{16}$$

where the i subscript refers to the i-th lamina. Explicitely Eq's.(16) are,

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix}_i = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} \\ \bar{Q}_{21} & \bar{Q}_{22} & \bar{Q}_{26} \\ \bar{Q}_{61} & \bar{Q}_{62} & \bar{Q}_{66} \end{bmatrix}_i \begin{bmatrix} \bar{e}_1 \\ \bar{e}_2 \\ \gamma_6 \end{bmatrix}_i - \theta_i \begin{bmatrix} \bar{\alpha}_1 \\ \bar{\alpha}_2 \\ \bar{\alpha}_6 \end{bmatrix}_i \tag{17}$$

where, upon using $c_i = \cos\phi_i$ and $s_i = \sin\phi_i$, we obtain,

$$\begin{aligned}
(\bar{Q}_{11})_i &= [Q_{11}c^4 + 2(Q_{12}+2Q_{66})s^2c^2 + Q_{22}s^4]_i \\
(\bar{Q}_{12})_i &= [(Q_{11}+Q_{22}-4Q_{66})s^2c^2 + Q_{12}(s^4+c^4)]_i \\
(\bar{Q}_{22})_i &= [Q_{11}s^4 + 2(Q_{12}+2Q_{66})s^2c^2 + Q_{22}c^4]_i \\
(\bar{Q}_{16})_i &= [(Q_{11}-Q_{12}-2Q_{66})sc^3 + (Q_{12}-Q_{22}+2Q_{66})s^3c]_i \\
(\bar{Q}_{26})_i &= [(Q_{11}-Q_{12}-2Q_{66})s^3c + (Q_{12}-Q_{22}+2Q_{66})sc^3]_i \\
(\bar{Q}_{66})_i &= [(Q_{11}+Q_{22}-2Q_{12}-2Q_{66})s^2c^2 + Q_{66}(s^4+c^4)]_i
\end{aligned} \tag{18}$$

We note from Eq's.(18) that the stiffness matrix of a lamina with respect to the laminate coordinate system is a full 3x3 matrix. However, there are still only four independent coefficients; any two of the $\bar{Q}_{jk}$ coefficients may be written as functions of the remaining four coefficients. The $\bar{\underset{\approx}{Q}}$ matrix is symmetric, that is, $\bar{Q}_{12} = \bar{Q}_{21}$, $\bar{Q}_{16} = \bar{Q}_{61}$, and $\bar{Q}_{26} = \bar{Q}_{62}$.

The fact that $\bar{\underset{\approx}{Q}}$ is,in general, full, means that coupling between normal and shear behavior occurs. In contrast, for isotropic materials, normal and shear behaviors uncouple. For cross ply laminates, that is laminates with orthogonal plies, subjected to inplane forces and bending moments, normal and shear behaviors uncouple. For angle ply laminates, that is laminates with some non-orthogonal plies, coupling between normal and shear behaviors occurs. This difference in behavior between cross ply laminates and angle ply laminates is shown by an illustrative example in section 11.

## 6.0 Strain-Displacement Relations for a Laminate

In accordance with common thin plate theory, the assumption that planes before loading remain planes after loading (Kirchhoff-Love theory) is taken as the basis of deformation of a laminate. In this case, the displacement field is given by,

$$\left.\begin{aligned} \bar{u} &= \bar{u}_o - z(\partial\bar{w}/\partial\bar{x}) \\ \bar{v} &= \bar{v}_o - z(\partial\bar{w}/\partial\bar{y}) \end{aligned}\right\} \qquad (19)$$

where $\bar{u}$ and $\bar{v}$ are the displacements in the $\bar{x}$ and $\bar{y}$ directions, $\bar{u}_o$ and $\bar{v}_o$ are the midplane displacements, $\bar{w}$ is the displacement in the $\bar{z}$ direction, and $\bar{z}$ is the distance from the midplane to the point of interest.

Using the linear strain-displacement relations,

$$\left.\begin{aligned} \bar{e}_x &= (\partial\bar{u}/\partial\bar{x}) \qquad e_y = (\partial\bar{v}/\partial\bar{y}) \\ \bar{\gamma}_{xy} &= (\partial\bar{u}/\partial\bar{y}) + (\partial\bar{v}/\partial\bar{x}) \end{aligned}\right\} \qquad (20)$$

equations (19) become,

$$\left.\begin{aligned}\bar{e}_1 &= (\partial\bar{u}_o/\partial\bar{x}) - \bar{z}(\partial^2\bar{w}/\partial\bar{x}^2)\\ \bar{e}_2 &= (\partial\bar{v}_o/\partial\bar{y}) - \bar{z}(\partial^2\bar{w}/\partial\bar{y}^2)\\ \bar{\gamma}_6 &= (\partial\bar{u}_o/\partial\bar{y} + \partial\bar{v}_o/\partial\bar{x}) - 2\bar{z}(\partial^2\bar{w}/\partial\bar{x}\partial\bar{y})\end{aligned}\right\} \tag{21}$$

Equations (21) may be written in vector form as,

$$\bar{\underset{\sim}{e}} = \bar{\underset{\sim}{e}}_o + \bar{z}\bar{\underset{\sim}{\kappa}} \tag{22}$$

where $\bar{\underset{\sim}{e}}_o$ is the strain vector associated with midplane extensional displacements given by

$$\left.\begin{matrix}(\bar{e}_1)_o\\ (\bar{e}_2)_o\\ (\bar{\gamma}_6)_o\end{matrix} \quad = \quad \begin{matrix}\partial\bar{u}_o/\partial\bar{x}\\ \partial\bar{v}_o/\partial\bar{y}\\ \partial\bar{u}_o/\partial\bar{y} + \partial\bar{v}_o/\partial\bar{x}\end{matrix}\right\} \tag{23}$$

and $\bar{\underset{\sim}{\kappa}}$ is the curvature vector associated with bending of the midplane, given by

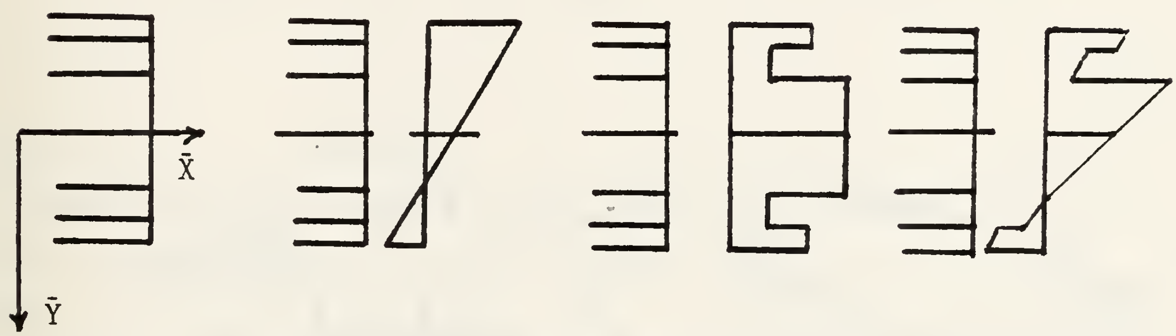
$$\left.\begin{matrix}(\bar{\kappa}_1)\\ (\bar{\kappa}_2)\\ (\bar{\kappa}_6)\end{matrix} \quad = \quad \begin{matrix}-\partial^2\bar{w}/\partial^2\bar{x}\\ -\partial^2\bar{w}/\partial^2\bar{y}\\ -2\partial^2\bar{w}/\partial\bar{x}\partial\bar{y}\end{matrix}\right\} \tag{24}$$

Since the midplane strains and curvatures belong to the laminate, they do not change from one lamina to another, and therefore the bar over these quantities is not required and henceforth will be dispensed with.

## 7.0 Force-Displacement Relations for a Laminate

Equation (22) shows that the strain at a point is the sum of two components, (i) midplane extensional strain, and (ii) midplane curvature. Figure 4. depicts the displacement, strain, modulus, and stress field with respect to the $\bar{x},\bar{z}$ plane at an arbitrary $\bar{y}$ location. The displacement and

strain fields are shown together in Fig.4b, while in fact, one is a scalar times the other. In accordance with the Kirchhoff-Love assumption, the displacement field is linear across the laminate thickness. As a result of the jump discontinuity of modulus from lamina to lamina (Fig.4c), the stress field also has a jump discontinuity from lamina to lamina (Fig.4d).




| Laminate Field | Displacement (and Strain) Field | Modulus Field | Stress Field |
|---|---|---|---|
| (a) | (b) | (c) | (d) |

Figure 4. Displacement, Strain, Modulus and Stress Distributions

The laminate forces $\underset{\sim}{\bar{N}}$ and moments $\underset{\sim}{\bar{M}}$ associated with a laminate are obtained by integrations of the stress field over the laminate thickness. The extensional laminate forces are given by,

$$\begin{bmatrix} \bar{N}_1 \\ \bar{N}_2 \\ \bar{N}_6 \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix} d\bar{z} \quad \text{or} \quad \underset{\sim}{\bar{N}} = \int \underset{\sim}{\bar{\sigma}} d\bar{z} \qquad (25)$$

and the laminate moments are given by,

$$\begin{bmatrix} \bar{M}_1 \\ \bar{M}_2 \\ \bar{M}_6 \end{bmatrix} = \int_{-\frac{t}{2}}^{\frac{t}{2}} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix} \bar{z}d\bar{z} \qquad \bar{\underset{\sim}{M}} = \int \bar{\underset{\sim}{\sigma}}\bar{z}d\bar{z} \tag{26}$$

where t is the plate thickness. The laminate forces and moments are shown in Figure 5. Components $N_1$ and $N_2$ are the normal components of force, and $N_6$ is the shear component. $M_1$ and $M_2$ are bending moments, and $M_6$ is the twisting moment.

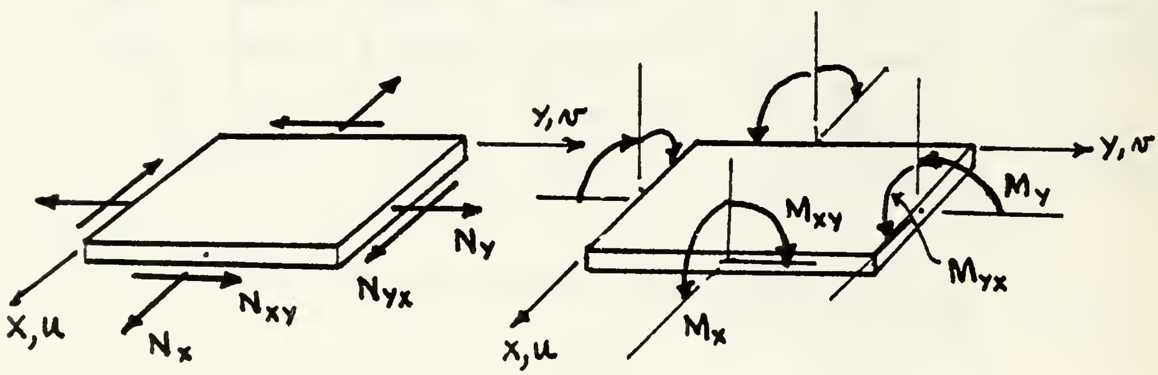



Figure 5. Laminate Forces and Moments

Since the stress field across the laminate thickness is the sum of the lamina stresses,

$$\bar{\underset{\sim}{\sigma}} = \sum_{i=1}^{n} \bar{\underset{\sim}{\sigma}}_i$$

equations (25) and (26) become

$$\bar{\underset{\sim}{N}} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \bar{\underset{\sim}{\sigma}}_i d\bar{z} \tag{27}$$

and

$$\bar{\underset{\sim}{M}} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \bar{\underset{\sim}{\sigma}}_i z d\bar{z} \tag{28}$$

where $z_{i-1}$ and $z_i$ are the upper and lower limits of integration for the i-th lamina, and n is the number of lamina comprising the laminate.

Substituting the expressions for lamina stresses in terms of strains, Eq's.(16), into the previous Eq's.(27) and (28) gives the laminate forces and moments in terms of strains as,

$$\bar{\underset{\sim}{N}} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_i - \bar{\underset{\sim}{\alpha}}_i \theta_i) d\bar{z} \tag{29}$$

and

$$\bar{\underset{\sim}{M}} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_i - \bar{\underset{\sim}{\alpha}}_i \theta_i) \bar{z} d\bar{z} \tag{30}$$

The strain in the i-th lamina is given by Eq.(22),

$$\bar{\underset{\sim}{e}}_i = \bar{\underset{\sim}{e}}_o + \bar{z} \bar{\underset{\sim}{\kappa}} \tag{31}$$

where $\bar{\underset{\sim}{e}}_o$ and $\bar{\underset{\sim}{\kappa}}$ are the midplane extensional strain and midplane curvature vectors respectively, and are constant from one lamina to another. Substituting Eq.(31) into Eq's.(29) and (30) gives,

$$\bar{\underset{\sim}{N}} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_o + \bar{z}\bar{\underset{\sim}{\kappa}} - \bar{\underset{\sim}{\alpha}}_i \theta_i) d\bar{z}$$

$$\bar{\underset{\sim}{M}} = \sum_{i=1}^{n} \int_{z_{i-1}}^{z_i} \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_o + \bar{z}\bar{\underset{\sim}{\kappa}} - \bar{\underset{\sim}{\alpha}}_i \theta_i) \bar{z} d\bar{z}$$

Since $\bar{\underset{\sim}{e}}_o$ and $\bar{\underset{\sim}{\kappa}}$ do not vary with $\bar{z}$, and $\bar{\underset{\approx}{Q}}_i, \bar{\underset{\sim}{\alpha}}_i$, and $\theta_i$ are constant for a lamina, the previous equations become,

$$\bar{\underset{\sim}{N}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i [(\bar{\underset{\sim}{e}}_o - \bar{\underset{\sim}{\alpha}}_i \theta_i) \int_{z_{i-1}}^{z_i} d\bar{z} + \bar{\underset{\sim}{\kappa}} \int_{z_{i-1}}^{z_i} \bar{z} d\bar{z}]$$

$$\bar{\underset{\sim}{M}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i [(\bar{\underset{\sim}{e}}_o - \bar{\underset{\sim}{\alpha}}_i \theta_i) \int_{z_{i-1}}^{z_i} \bar{z} d\bar{z} + \bar{\underset{\sim}{\kappa}} \int_{z_{i-1}}^{z_i} \bar{z} d\bar{z}]$$

Upon integration, we obtain,

$$\bar{\underset{\sim}{N}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_o - \bar{\underset{\sim}{\alpha}}_i \theta_i)(\bar{z}_i - \bar{z}_{i-1}) + \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i \bar{\underset{\sim}{\kappa}}_i \left(\frac{\bar{z}_i^2 - \bar{z}_{i-1}^2}{2}\right) \tag{32}$$

and

$$\bar{\underset{\sim}{M}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_o - \bar{\underset{\sim}{\alpha}}_i\theta_i)\frac{(\bar{z}_i^2 - \bar{z}_{i-1}^2)}{2} + \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i\bar{\underset{\sim}{\kappa}}_i\frac{(\bar{z}_i^3 - \bar{z}_{i-1}^3)}{6} \qquad (33)$$

Defining the matrices,

$$\bar{\underset{\approx}{A}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i(\bar{z}_i - \bar{z}_{i-1}) \qquad (34)$$

$$\bar{\underset{\approx}{B}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i(\bar{z}_i^2 - \bar{z}_{i-1}^2)/2 \qquad (35)$$

$$\bar{\underset{\approx}{D}} = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i(\bar{z}_i^3 - \bar{z}_{i-1}^3)/6 \qquad (36)$$

and vectors,

$$\bar{\underset{\sim}{N}}_T = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i\bar{\underset{\sim}{\alpha}}_i\theta_i(\bar{z}_i - \bar{z}_{i-1}) \qquad (37)$$

$$\bar{\underset{\sim}{M}}_T = \sum_{i=1}^{n} \bar{\underset{\approx}{Q}}_i\bar{\underset{\sim}{\alpha}}_i\theta_i(\bar{z}_i^2 - \bar{z}_{i-1}^2)/2 \qquad (38)$$

equations (32) and (33) become,

$$\bar{\underset{\sim}{N}} = \bar{\underset{\approx}{A}}\bar{\underset{\sim}{e}}_o + \bar{\underset{\approx}{B}}\bar{\underset{\approx}{\kappa}} - \bar{\underset{\sim}{N}}_T \qquad (39)$$

$$\bar{\underset{\sim}{M}} = \bar{\underset{\approx}{B}}\bar{\underset{\sim}{e}}_o + \bar{\underset{\approx}{D}}\bar{\underset{\approx}{\kappa}} - \bar{\underset{\sim}{M}}_T \qquad (40)$$

Equations (39) and (40) are the "force-displacement" relations for a laminate.

The $\bar{\underset{\sim}{N}}_T$ and $\bar{\underset{\sim}{M}}_T$ vectors are the so called thermal force vector and thermal moment vector respectively, as they result from a thermal environment. If the temperature of each lamina is equal to the stress free temperature (i.e., the curing temperature), then $\theta_i$ is zero for each lamina, and the thermal vectors are equal to the zero vector. In this case, Eq's. (39) and (40) reduce to the force-displacement relations for a laminate not subjected to thermal effects.

The $\bar{\underset{\approx}{A}}$ matrix relates inplane forces to inplane strains, and the $\bar{\underset{\approx}{D}}$ matrix relates moments to curvature. The $\bar{\underset{\approx}{B}}$ matrix

couples inplane forces to curvature, as well as moments to inplane strains. In the special, albeit common, case of symmetric laminates, the contribution to $\underset{\approx}{\bar{B}}$ from a lamina above the midplane is cancelled by the contribution from a counterpart lamina below the midplane, and as a result, the $\underset{\approx}{\bar{B}}$ matrix is the zero matrix, i.e., $\underset{\approx}{\bar{B}} = \underset{\approx}{0}$. Thus for symmetric laminates,

$$\underset{\sim}{\bar{N}}_T = \underset{\approx}{\bar{A}}\,\underset{\sim}{\bar{e}}_o - \underset{\sim}{\bar{N}}_T \tag{41}$$

$$\underset{\sim}{\bar{M}}_T = \underset{\approx}{\bar{D}}\,\underset{\sim}{\bar{\kappa}} - \underset{\sim}{\bar{M}}_T \tag{42}$$

and therefore the extensional and bending behaviors are uncoupled. If the laminate is symmetric and the temperature distribution is symmetric with respect to the midplane, then from Eq.(38), the thermal moment vector is equal to the zero vector and Eq.(42) reduces to

$$\underset{\sim}{\bar{M}} = \underset{\approx}{\bar{D}}\,\underset{\sim}{\bar{\kappa}} \tag{43}$$

## 8.0 Stress-Force Relations for a Lamina in a Symmetric Laminate

In order to determine whether a ply has failed, it is necessary to determine the stresses in each ply. The developement of the stress-force relations for a lamina, which follows, is restricted to symmetric laminates. This is not a severe restriction since symmetric laminates are in most common use. Solving Eq's.(41) and (42) for strains and curvatures gives,

$$\underset{\sim}{\bar{e}}_o = \underset{\approx}{\bar{A}}^{-1}(\underset{\sim}{\bar{N}} + \underset{\sim}{\bar{N}}_T) \tag{44}$$

and

$$\underset{\sim}{\bar{\kappa}} = \underset{\approx}{\bar{D}}^{-1}(\underset{\sim}{\bar{M}} + \underset{\sim}{\bar{M}}_T) \tag{45}$$

The lamina strains are related to the midplane strains and curvatures of the laminate by Eq.(31),

$$\underset{\sim}{\bar{e}}_i = \underset{\sim}{\bar{e}}_o + \bar{z}\,\underset{\sim}{\bar{\kappa}} \tag{31}$$

Substitution of Eq's.(44) and (45) into Eq.(31) gives,

$$\bar{\underset{\sim}{e}}_i = \bar{\underset{\approx}{A}}^{-1}(\bar{\underset{\sim}{N}} + \bar{\underset{\sim}{N}}_T) + \bar{z}_i\bar{\underset{\approx}{D}}^{-1}(\bar{\underset{\sim}{M}} + \bar{\underset{\sim}{M}}_T) \tag{46}$$

where $\bar{z}_i$ is the distance from the laminate midplane to the middle of the i-th lamina, that is,

$$\bar{z}_i = (z_i + z_{i-1})/2 \tag{47}$$

Lamina stresses with respect to laminate axes are related to lamina strains in laminate axes by Eq.(16) as,

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\approx}{Q}}_i(\bar{\underset{\sim}{e}}_i - \bar{\underset{\sim}{\alpha}}_i\theta_i) \tag{16}$$

Finally then, substitution of Eq.(46) into Eq.(16) gives the lamina stresses, with respect to laminate axes, in terms of laminate inplane forces $\bar{\underset{\sim}{N}}$, laminate moments $\bar{\underset{\sim}{M}}$, thermal forces $\bar{\underset{\sim}{N}}_T$, and thermal moments $\bar{\underset{\sim}{M}}_T$ as,

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\approx}{Q}}_i[\bar{\underset{\approx}{A}}^{-1}(\bar{\underset{\sim}{N}} + \bar{\underset{\sim}{N}}_T) + \bar{z}_i\bar{\underset{\approx}{D}}^{-1}(\bar{\underset{\sim}{M}} + \bar{\underset{\sim}{M}}_T) - \bar{\underset{\sim}{\alpha}}_i\theta_i] \tag{48}$$

Equation (48) takes on special forms for particular cases; a few of which are presented below.

a) Temperature free effects occur when the plate is at a uniform temperature equal to the curing temperature of laminate. In this case, $\theta_i = 0$, and therefore $\bar{\underset{\sim}{N}}_T$ and $\bar{\underset{\sim}{M}}_T$ are equal to the zero vector, and Eq.(48) reduces to,

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\approx}{Q}}_i(\bar{\underset{\approx}{A}}^{-1}\bar{\underset{\sim}{N}} + \bar{z}_i\bar{\underset{\approx}{D}}^{-1}\bar{\underset{\sim}{M}}) \tag{49}$$

In the absence of lateral loads on the laminate, $\bar{\underset{\sim}{M}} = \underset{\sim}{0}$, and Eq.(49) becomes,

$$\bar{\underset{\sim}{\sigma}} = \bar{\underset{\approx}{Q}}\bar{\underset{\approx}{A}}^{-1}\bar{\underset{\sim}{N}} \tag{50}$$

b) Absence of moments occurs when the laminate is not subjected to lateral loads. In this case, $\bar{\underset{\sim}{M}} = \underset{\sim}{0}$, and

equation (48) reduces to,

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\approx}{Q}}_i[\bar{\underset{\approx}{A}}^{-1}(\bar{\underset{\sim}{N}} + \bar{\underset{\sim}{N}}_T) + \bar{z}_i\bar{\underset{\approx}{D}}^{-1}\bar{\underset{\sim}{M}}_T - \bar{\underset{\sim}{\alpha}}_i\theta_i] \tag{51}$$

If the temperature field is symmetric with respect to the midplane of the laminate, then $\bar{\underset{\sim}{M}}_T$ is the zero vector and Eq.(51) becomes,

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\sim}{Q}}_i[\bar{\underset{\sim}{A}}^{-1}(\bar{\underset{\sim}{N}} + \bar{\underset{\sim}{N}}_T) - \bar{\underset{\sim}{\alpha}}_i\theta_i] \tag{52}$$

c) <u>Thermal effects only</u> occur when the laminate is not subjected to inplane and lateral loads. In this case $\bar{\underset{\sim}{N}} = \bar{\underset{\sim}{M}} = \underset{\sim}{0}$, and Eq.(48) reduces to

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\approx}{Q}}_i[\bar{\underset{\approx}{A}}^{-1}\bar{\underset{\sim}{N}}_T + \bar{z}_i\bar{\underset{\approx}{D}}^{-1}\bar{\underset{\sim}{M}}_T - \bar{\underset{\sim}{\alpha}}_i\theta_i] \tag{53}$$

If the temperature field is symmetric with respect to the laminate midplane, then $\bar{\underset{\sim}{M}}_T = \underset{\sim}{0}$, and Eq.(53) gives,

$$\bar{\underset{\sim}{\sigma}}_i = \bar{\underset{\approx}{Q}}_i[\bar{\underset{\approx}{A}}^{-1}\bar{\underset{\sim}{N}}_T - \bar{\underset{\sim}{\alpha}}_i\theta_i] \tag{54}$$

## 9.0 <u>Failure Criterion</u>

When a composite plate is subjected to a load and/or thermal environment, stresses develope in each lamina in accordance with Eq.(48). Depending upon the magnitude of the lamina stresses, one or more lamina may fail. When a lamina fails, there are two effects. First, if we assume brittle failure, then a lamina will relinquish it's load carrying capacity upon failure, and there will be a redistribution of load to the remaining non failed lamina. Secondly, when a lamina fails, the stiffness matrices $\bar{\underset{\approx}{A}}$ and $\bar{\underset{\approx}{D}}$ change in accordance with Eq's.(34) and (36), and thus the redistribution of stresses in the non failed lamina is not simply a change in magnitude, but rather a change in the stress state.

The hypothesis of a brittle failure of a lamina is an

idealization of the actual behavior. Actually, lamina have some ductility and therefore continue to carry some fraction of load, and contribute some stiffness to the laminate upon failure. Here we assume a failed lamina serves no useful purpose after it's failure.

As will be demonstrated by illustrative examples in section 11, failure of a ply does not necessarily mean that the laminate has failed. There we will show that a cross ply laminate subjected to uniaxial loading does not fail upon failure of the first ply, and therefore the magnitude of the load can be increased beyond first ply failure. We also show that under uniaxial loading, angle ply laminates fail when the first ply fails.

There is no way to establish the validity of any failure criterion on theoretical grounds. All that can be done is to show that a failure criterion gives analytical results which agree, for a wide class of problems, with experimental results. There are several failure theories which give good results and are in common use. The most general of the macroscopic failure criteria, which includes many of the other criteria as special cases, is the Tsai-Wu criterion, Ref.(2). It is the failure criterion adopted in this work.

According to the Tsai-Wu criterion, a lamina fails when the general quadratic equation (in tensor form),

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1 \qquad i,j=1,2,,,6 \tag{55}$$

is satisfied. In Eq.(55), $F_i$ and $F_{ij}$ are strength tensors of second and fourth order, and the equation is in contracted notation. In the case of plane stress, the stress state for composite plates, Eq.(55) in terms of natural coordinates of a lamina, becomes

$$F_{11}\sigma_1^2 + 2F_{12}\sigma_1\sigma_2 + F_{22}\sigma_2^2 + F_{66}\sigma_6^2 + F_1\sigma_1 + F_2\sigma_2 = 1 \tag{56}$$

Equation (56) is applied to each lamina. If the left hand

side of Eq.(56) is less than unity, then the lamina has not failed. If the left hand side is greater than unity, then the lamina has failed. Failure of a lamina is imminent (impending) when the left hand side is equal to unity, i.e., when Eq.(56) is satisfied identically.

In order to use Eq.(56) directly, that is without modification, it is necessary to use lamina stresses with respect to lamina axes. The use of lamina stresses with respect to laminate axes would require the transformation of the second and fourth order tensors, $F_i$ and $F_{ij}$, to laminate axes. Although this could be done, it is more expeditious to transform the lamina stresses with respect to laminate axes, given by Eq's.(48), into lamina stresses with respect to lamina axes. The transformation is accomplished by the use of Eq.(14). Thus, we obtain,

$$\begin{aligned}(\sigma_1)_i &= (\bar{\sigma}_1)_i c_i^2 + (\bar{\sigma}_2)_i s_i^2 + 2(\bar{\sigma}_6)_i c_i s_i \\ (\sigma_2)_i &= (\bar{\sigma}_1)_i s_i^2 + (\bar{\sigma}_2)_i c_i^2 - 2(\bar{\sigma}_6)_i c_i s_i \qquad (57) \\ (\sigma_6)_i &= -(\bar{\sigma}_1) c_i s_i + (\bar{\sigma}_2)_i c_i s_i + (\bar{\sigma}_6)_i (c_i^2 - s_i^2)\end{aligned}$$

where $c_i = \cos\phi_i$, $s_i = \sin\phi_i$, and i denotes the lamina.

Five of the six strength parameters of Eq.(56) may be determined by a series of simple experiments as follows. A tension test of a lamina along it's fiber direction gives the longitudinal tension strength, X. A compression test along the fiber axis gives the longitudinal compressive strength, X'. Tension and compressive tests in the transverse direction (i.e., orthogonal to the fiber), gives the transverse tension and compressive strengths, Y and Y', respectively. A shear test of the lamina gives it's shear strength, S. The strength parameters in Eq.(56) are related to the experimentally determined strengths as follows. Applying Eq.(56) to each of the five stress states associated with the five tests gives five simultaneous equations which may

be solved for $F_{11}, F_{22}, F_{66}, F_1$, and $F_2$. For example, Eq.(56) for the longitudinal tension test becomes,

$$F_{11}X^2 + F_1X = 1$$

and for the longitudinal compression test, Eq.(56) is,

$$F_{11}X'^2 - F_1X' = 1$$

Solving these equations for $F_{11}$ and $F_1$ gives the first two of Eq's.(58). The remaining strength parameters are obtained in a similar way.

$$\begin{aligned} F_{11} &= (1/XX') \\ F_1 &= (1/X) - (1/X') \\ F_{22} &= (1/YY') \\ F_2 &= (1/Y) - (1/Y') \\ F_{66} &= (1/S^2) \end{aligned} \tag{58}$$

The remaining strength parameter $F_{12}$ is obtained on mathematical grounds. In crder for the failure to be a closed curve, the quadratic equation must be an ellipse. This leads to the condition (see Ref.(3) for details),

$$-1 < F_{12}^2/F_{11}F_{22} < 1$$

If the Tsai-Wu criterion is required to be a generalization of the von Mises criterion, then $(F_{12}^2/F_{11}F_{22})$ must be equal to -0.5. This condition leads to,

$$F_{12} = -0.5(F_{11}F_{22})^{1/2} \tag{59}$$

Some typical values of lamina strengths are presented in Appendix A.

## 10.0 Strength Ratio and Failure Analysis

As discussed in the previous section, the Tsai-Wu criterion of Eq.(56) can be used to determine whether or not a

ply has failed by evaluation of the left hand side of the equation. By introducing a parameter, called the strength ratio, the criterion can be used to determine when a ply fails, the sequence of ply failure, and the load for laminate failure.

Given the temperature field of a laminate, $T_i$, and the relative magnitudes of the components of the laminate "load" vectors, $\bar{\tilde{N}}$ and $\bar{\tilde{M}}$, we seek to determine the allowable load vectors, $\bar{\tilde{N}}_a$ and $\bar{\tilde{M}}_a$, which will bring a lamina to imminent failure. Since the laminate stiffness matrices, $\bar{\approx{A}}$ and $\bar{\approx{D}}$, change with every ply failure, a cyclic (iterative) procedure is required to trace the sequence of ply failure.

The m-th cycle determines the m-th ply which will fail. During the m-th cycle, the strength ratio of the i-th ply is defined as,

$$R_{i(m)} = \bar{N}_{1a}/\bar{N}_1 \tag{60}$$

This ratio is the same for all components of $\bar{\tilde{N}}$ and $\bar{\tilde{M}}$, that is, $\bar{N}_{ja}/\bar{N}_j = \bar{M}_{ja}/\bar{M}_j$ , for j=1,2,6. The allowable stresses in a lamina during the m-th cycle, with respect to laminate axes, are given by Eq's.(48),

$$(\bar{\tilde{\sigma}}_{ia})_{(m)} = \bar{\approx{Q}}_i [\bar{\approx{A}}^{-1}_{(m)} (\bar{\tilde{N}}_{a(m)} + \bar{\tilde{N}}_T) + \bar{z}_i \bar{\approx{D}}^{-1}_{(m)} (\bar{\tilde{M}}_{a(m)} + \bar{\tilde{M}}_T) - \bar{\tilde{\alpha}}_i \tilde{\theta}_i] \tag{48}$$

where the 'a' subscript denotes allowable, the i subscript denotes the i-th lamina, the m subscript denotes the m-th cycle, and $\bar{\approx{A}}_{(m)}$ and $\bar{\approx{D}}_{(m)}$ are the stiffness matrices of the m-th cycle laminate. In Eq.(48), i takes on the values of only the non failed lamina. The m-th cycle laminate is comprised of the lamina which have not failed during the previous (m-1) cycles.

It is convenient to partition the lamina stresses into two parts; one part due to the $\bar{\tilde{N}}$ and $\bar{\tilde{M}}$ loads, and another part due to the $\bar{\tilde{N}}_T$ and $\bar{\tilde{M}}_T$ thermal vectors. Denoting the

stresses due to loads by $\bar{\underset{\sim}{\sigma}}_{iL}$, and the stresses due to temperature by $\bar{\underset{\sim}{\sigma}}_{iT}$, we have,

$$\bar{\underset{\sim}{\sigma}}_{ia(m)} = \bar{\underset{\sim}{\sigma}}_{iL(m)} + \bar{\underset{\sim}{\sigma}}_{iT(m)} \tag{61}$$

where the load stresses $\bar{\underset{\sim}{\sigma}}_{iL(m)}$, and thermal stresses $\bar{\underset{\sim}{\sigma}}_{iT(m)}$ are given by,

$$\bar{\underset{\sim}{\sigma}}_{iL(m)} = \bar{\underset{\approx}{Q}}_i [\, \bar{\underset{\approx}{A}}^{-1}_{(m)} \cdot \bar{\underset{\sim}{N}}_{a(m)} + \bar{z}_i \bar{\underset{\approx}{D}}^{-1}_{(m)} \cdot \bar{\underset{\sim}{M}}_{a(m)} ] \tag{62}$$

$$\bar{\underset{\sim}{\sigma}}_{iT(m)} = \bar{\underset{\approx}{Q}}_i [\, \bar{\underset{\approx}{A}}^{-1}_{(m)} \cdot \bar{\underset{\sim}{N}}_{T} + \bar{z}_i \bar{\underset{\approx}{D}}^{-1}_{(m)} \cdot \bar{\underset{\sim}{M}}_{T} ] \tag{63}$$

Using Eq's.(57), the above lamina stresses with respect to laminate axes are transformed into lamina stresses with respect to lamina axes for use in Eq.(56). Substituting lamina stresses with respect to lamina axes into the Tsai-Wu criterion of Eq.(56) gives a quadratic equation for the i-th lamina and m-th cycle,

$$a_{i(m)} R^2_{i(m)} + b_{i(m)} R_{i(m)} + c_{i(m)} = 0 \tag{64}$$

where the coefficients are,

$$a_{i(m)} = [F_{11}\sigma^2_{1L} + 2F_{12}\sigma_{1L}\sigma_{2L} + F_{22}\sigma^2_{2L} + F_{66}\sigma^2_{6L}]_{i(m)} \tag{65}$$

$$b_{i(m)} = [2(F_{11}\sigma_{1L}\sigma_{1T} + F_{12}\sigma_{1L}\sigma_{2T} + F_{12}\sigma_{2L}\sigma_{1T} + F_{22}\sigma_{2L}\sigma_{2T} + F_{66}\sigma_{6L}\sigma_{6T}) + F_1\sigma_{1L} + F_2\sigma_{2L}]_{i(m)} \tag{66}$$

$$c_{i(m)} = [F_{11}\sigma^2_{1T} + 2F_{12}\sigma_{1T}\sigma_{2T} + F^2_{22}\sigma_{2T} + F_{66}\sigma_{6T} + F_1\sigma_{1T} + F_2\sigma_{2T}]_{i(m)} - 1 \tag{67}$$

The strength ratios $R_{i(m)}$ for each of the lamina comprising the m-th cycle laminate are obtained as the roots of Eq. (64). The lamina with the smallest absolute strength ratio is the lamina which will fail during the m-th cycle. Thus the load factor to bring the m-th ply to failure is,

$$R^*_{(m)} = \mathrm{Min}[\mathrm{Abs}(R_{i(m)})] \tag{68}$$

Laminate failure occurs when all lamina have failed. The strength ratio (or load factor) for the laminate to fail, denoted by $R_u$, is the maximum of the $R^*_{(m)}$ load factors, that is,

$$R_u = \mathrm{Max}[R^*_{(m)}] \tag{69}$$

If the maximum strength ratio is obtained for m<n, where n is the number of plies comprising the laminate, then the laminate fails after m plies have failed. The allowable loads that the laminate can sustain without failure, for a given temperature distribution, are obtained from Eq.(60) as,

$$\begin{aligned} \bar{\underset{\sim}{N}}_a &= R_u \bar{\underset{\sim}{N}} \\ \bar{\underset{\sim}{M}}_a &= R_u \bar{\underset{\sim}{M}} \end{aligned} \tag{70}$$

Illustrative examples are presented in section 11.

## 11.0 Results and Conclusions

An analysis for the prediction of the strength of fiber reinforced composite plates subjected to loads and thermal environment, described in this report, has resulted in a computer program for implementation of the analysis. A listing of the program, as well as directions for it's use, is given in appendices B and C.

Several analyses have been carried out in order to compare the results of the present analysis to other analytical results (ref.(4)), as well as experimental results. The failure criterion used in Ref.(4) is the Tsai-Hill criterion.

Figures 6 through 8 present some preliminary results. Figure 6 presents four sets of results for cross-ply laminates subjected to uniaxial loading:

1) experimental data for E-glass/epoxy from Ref.(4),
2) analytical results for E-glass/epoxy from Ref.(4),
3) analytical results for E-glass/epoxy from the present analysis, and
4) analytical results for T300/5208 graphite/epoxy from the present analysis.

The abscissa M, called the cross-ply ratio and defined by

$$M = \Sigma t_k / \Sigma t_j \quad \text{for even } j\text{, and odd } k \qquad (71)$$

is a measure of the relative quantity of 90 and 0 degree plies. It is seen that for cross ply laminates under uniaxial loading, there is excellent agreement between experimental data and analytical results for the E-glass/epoxy laminates. All results are for a curing temperature of 270 degrees Fahrenfeit, and a loading environment at 70 degrees. In each case, the laminate did not fail at first ply failure. The first ply failure load is given in parentheses below the laminate failure load.

Figure 7 presents the same set of results for angle ply laminates as Figure 6 does for cross-ply laminates. In this case, the laminate has the layup [$-\phi/+\phi$] with equal thickness plies. It can be seen that results from the present analysis for E glass/epoxy laminates are in good agreement with the experimental and analytical results of Ref. (4) for $\phi$ less than 30°, and $\phi$ greater than 60°. Tsai, Ref. (5), has a forthcoming report which shows that the Tsai-Wu criterion is in good agreement with experimental results for T300/5208 graphite/epoxy angle ply laminates.

Figure 8 shows the effect of temperature on laminate failure for $\pm 45°$ angle ply laminates under uniaxial loading for both E-glass/epoxy, and T300/5208 graphite epoxy laminates.

In order to gain an understanding of the effect of

temperature on laminate strength, a series of computer analyses will be undertaken. The results of this computational effort will be presented in another report.

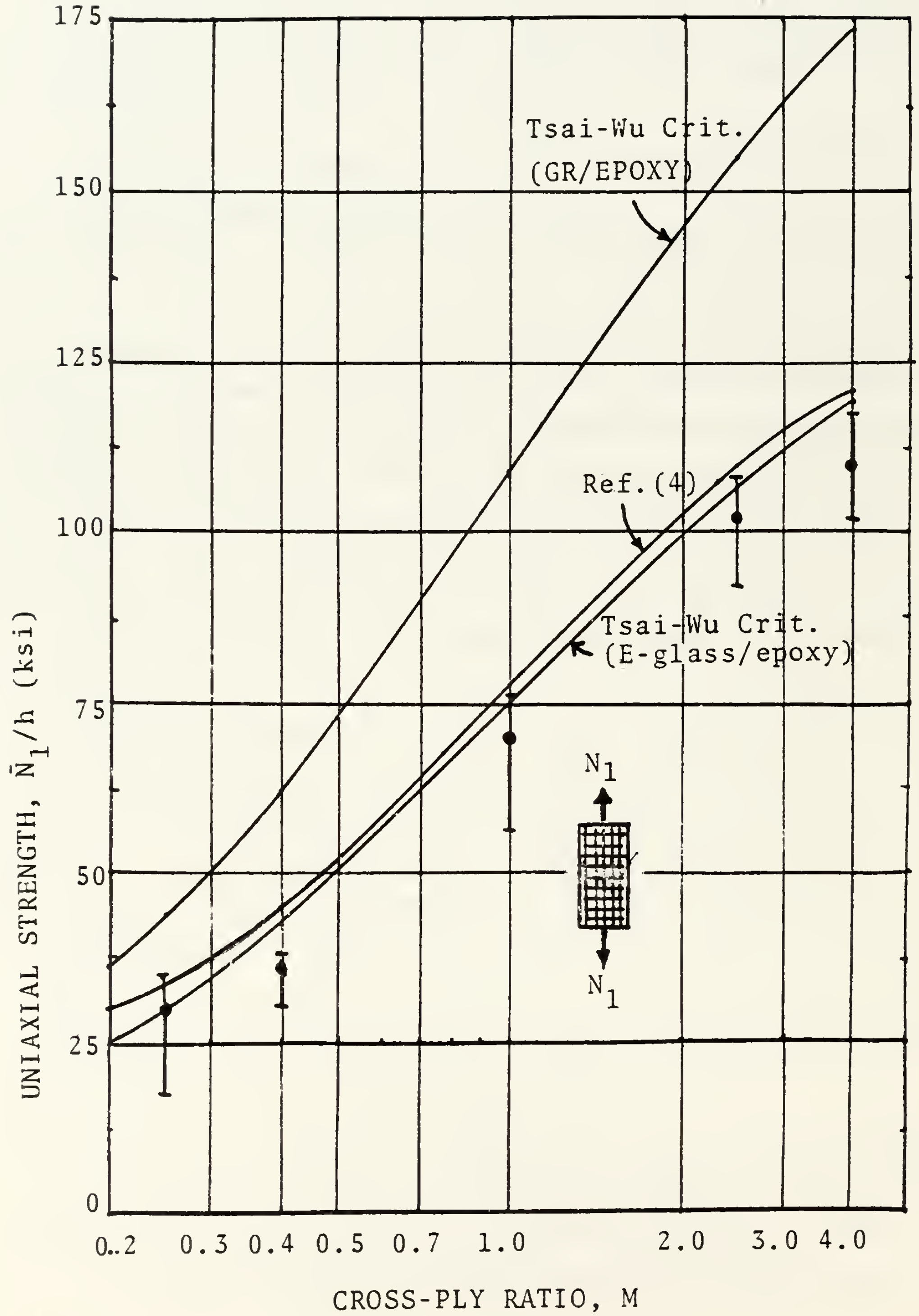



Figure 6. Experimental and Analytical Results for Cross-Ply Laminates

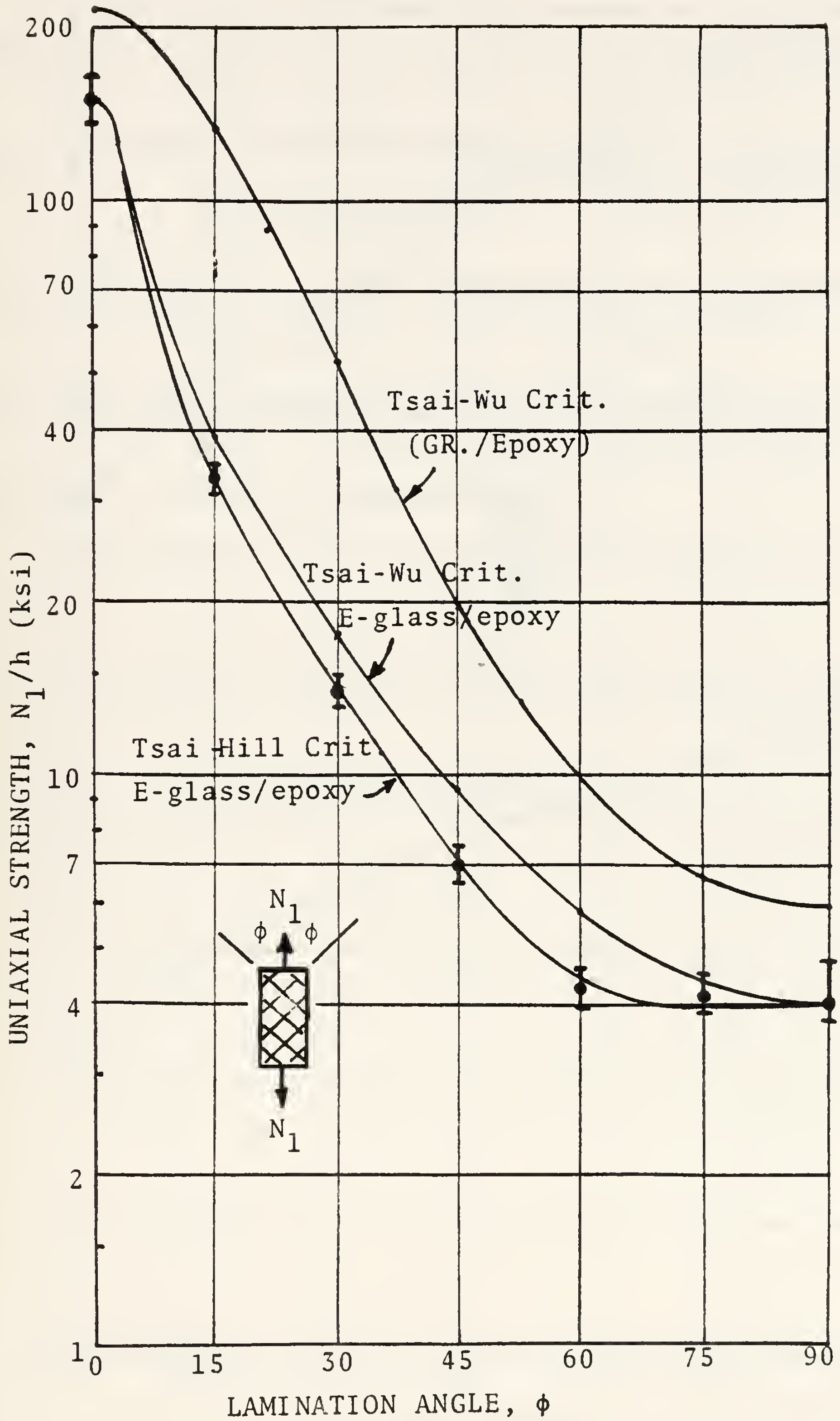



Figure 7. Experimental and Analytical Results for Angle-Ply Laminates

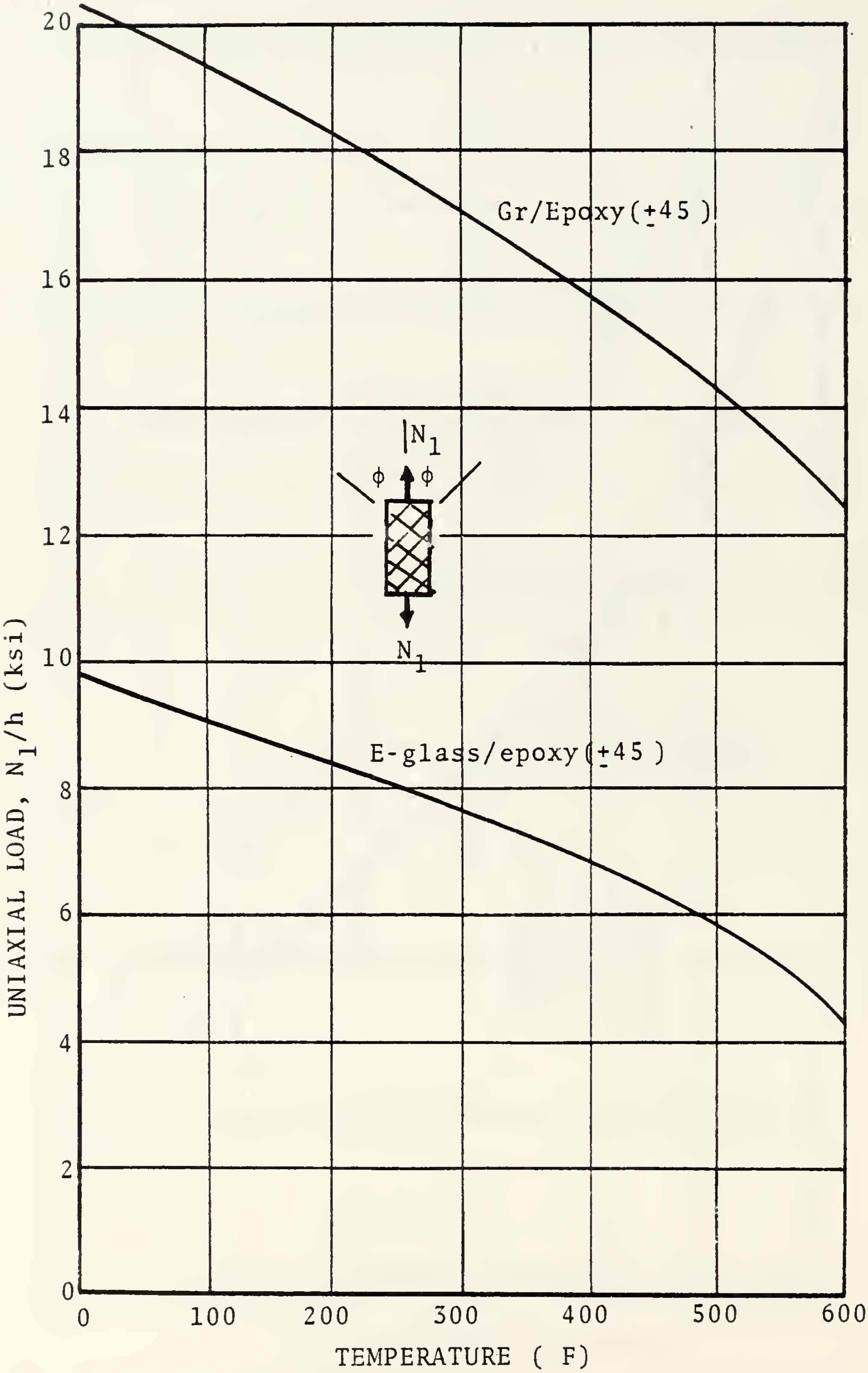



Figure.8 Load versus Temperature

# References


1. Jones, R.E.: Mechanics of Composite Materials. Mcgraw-Hill Book Co.,1975.
2. Tsai, S.W., and E.M. Wu: A General Theory of Strength for Anisotropic Materials, J.Comp. Materials, Jan. 1971, pp.58-80.
3. Tsai, S.W., and H.T. Hahn: Introduction to Composite Materials. Technion Publ. Co.,1980.
4. Tsai, S.W., D.F. Adams, and D.R. Doner: Analysis of Composite Structures, NASA CR-620, Nov.1966.
5. Tsai, S.W.: Private Communication

## Appendix A

### Typical Properties for Two Lamina

The table below presents some properties for two lamina, E-glass/epoxy, and graphite/epoxy. The information is taken from references (3) and (4).

| Property | Units SI | (Engr'g.) | E-glass/epoxy | | graphite/epoxy | |
|---|---|---|---|---|---|---|
| $E_x$ | GPa | ($10^6$psi) | 53.8 | (7.8) | 181. | (26.2) |
| $E_y$ | GPa | ($10^6$psi) | 17.9 | (2.6) | 10.3 | (1.49) |
| $\nu_{xy}$ | dimensionless | | 0.25 | (.25) | 0.28 | (.28) |
| $G_{xy}$ | GPa | ($10^6$psi) | 8.6 | (1.25) | 7.17 | (1.04) |
| * $\nu_{yx}$ | dimensionless | | .083 | (.083) | 0.16 | (0.16) |
| $\alpha_x$ | 1/C | (1/F) | 6.3μ | (3.5)μ | 0.02μ | (.011)μ |
| $\alpha_y$ | 1/C | (1/F) | 20.5 | (11.4)μ | 22.5μ | (12.5)μ |
| X | MPa | (ksi) | 1034. | (150) | 1500. | (217.5) |
| X' | MPa | (ksi) | 1034. | (150) | 1500. | (217.5) |
| Y | MPa | (ksi) | 27.6 | (4.) | 40. | (5.8) |
| Y' | MPa | (ksi) | 138. | (20.) | 246. | (35.7) |
| S | MPa | (ksi) | 41.4 | (6.) | 68. | (9.9) |
| † $v_f$ | dimensionless | | | | 0.7 | (0.7) |

* calculated by EQ.(12)

† volume fraction of fiber

μ denotes micro (i.e., $10^{-6}$)

# Appendix B

## COMPUTER LISTING

```
C     ****************************************************************     STR00
C     *                                                              *     STR00
C     *    ANALYSIS OF THE STRENGTH OF A SYMMETRIC COMPOSITE         *     STR00
C     *       PLATE DUE TO A THERMAL ENVIRONMENT                     *     STR00
C     *                                                              *     STR00
C     ****************************************************************     STR00
C                                                                          STR00
C     THIS IS THE MAIN PROGRAM WHICH READS IN THE PROPERTIES OF THE        STR00
C     COMPOSITE, AS WELL AS THE INPLANE LOADS AND THERMAL DISTRIBU-        STR00
C     TION THROUGH THE LAMINATE. IN ADDITION, THIS PROGRAM CALLS THE       STR00
C     SUBROUTINE CYCLE, WHICH IN TURN CALLS SUBROUTINE FAILUR.             STR00
C                                                                          STR00
      COMMON ANGLE(100),Z(101),A(3,3),D(3,3),AT(3,3),DT(3,3),AINV(3,3),WSTR00
     1K(50),DINV(3,3),PLYTHK(100),QBAR11(100),QBAR12(100),QBAR22(100),QBSTR00
     2AR16(100),QBAR26(100),QBAR66(100),ALPHA1(100),ALPHA2(100),ALPHA3(1STR00
     300),DELTEM(100),STRES1(100),STRES2(100),STRES3(100),TEM(100),STRESSTR00
     4X(100),STRESY(100),TAUXY(100),R1(100),R2(100),ZN1,ZN2,ZN3,FXX,FX,FSTR00
     5YY,FY,FSS,FXY,QXX,QXY,QYY,QSS,ALFAX,ALFAY,STFRTM,STRSL1(100),        STR00
     6STRSL2(100),STRSL3(100),STRST1(100),STRST2(100),STRST3(100),         STR00
     7STRSTX(100),STRSTY(100),TAUTXY(100),STRSLX(100),STRSLY(100),TAULXYSTR00
     8(100),R(70),RMAX,ZM1,ZM2,ZM3,                                        STR00
     9NPLY,ICYCLE,IFAIL,NPL,IT(100)                                        STR00
C                                                                          STR00
C     1 AND 2 AFTER A NAME DENOTE THE QUANTITY IS W.R.T. THE PLATE         STR00
C        AXES (I.E.,THE GLOBAL AXES OF THE SYSTEM).                        STR00
C     X AND Y DENOTE THE NATURAL COORDINATES OF A PLY. X IS ALONG THE      STR00
C        FIBER AND Y IS THE TRANSVERSE DIRECTION                           STR00
C     A AND D ARE THE INPLANE AND BENDING STIFFNESS MATRICES OF THE COMPO-STR00
C        SITE PLATE W.R.T. LAMINATE AXES. THEY CHANGE AS PLIES FAIL.       STR00
C     AINV AND DINV ARE THE INVERSES OF THE A AND D MATRICES, RESPECTIVELYSTR00
C     ALFAX, AND ALFAY ARE THE COEFFICIENTS OF THERMAL EXPANSION W.R.T.    STR00
C        THE NATURAL COORDINATES OF A PLY                                  STR00
C     ALPHA1(I),ALPHA2(I),ALPHA3(I) ARE THE COEFFICIENTS OF THERMAL        STR00
C        EXPANSION OF THE I-TH PLY W.R.T. THE LAMINATE AXES.               STR00
C     ANGLE(I) IS THE ANGLE BETWEEN THE FIBER DIRECTION (I.E.,X-AXIS) OF   STR00
C        THE I-TH PLY AND THE 1-AXIS OF THE LAMINATE COORDINATE SYSTEM.    STR00
C     AT AND DT ARE THE A AND D MATRICES DIVIDED BY THE PLATE THICKNESS    STR00
C     DELTEM(I) IS THE TEMPERATURE DIFFERENCE OF THE I-TH PLY, AND         STR00
C        IS EQUAL TO TEM(I)-STFRTM.                                        STR00
C     FXX,FX,FYY,FY,FSS ARE THE FAILURE PARAMETERS IN THE TSAI-WU FAILURE STR00
C        CRITERION AND ARE OBTAINED FROM X, XPRIME, Y, YPRIME, SS, AND     STR00
C        FXYSTR.                                                           STR00
C     IT(I) EQUALS 0 FOR NONFAILED PLIES, AND EQUALS 1 FOR FAILED PLIES    STR00
C     NPLY IS THE NUMBER OF PLIES ABOVE THE MIDPLANE OF A SYMMETRIC        STR00
C        LAMINATE. THE TOTAL NUMBER OF PLIES IS 2*NPLY. THE NUMBER OF      STR00
C        PLIES DECREASES DURING THE ANALYSIS AS PLIES FAIL.                STR00
C     PLYTHK(I) IS THE PLY THICKNESS OF THE I-TH PLY.                      STR00
C     QXX,QXY,QYY,AND QSS ARE THE STIFFNESS COEFFICIENTS OF A PLY W.R.T.   STR00
C        THE LAMINA X AND Y AXES.                                          STR00
C     QBAR11(I),QBAR12(I),QBAR22(I),QBAR16(I),QBAR26(I), AND QBAR66(I)     STR00
C        ARE THE STIFFNESS COEFFICIENTS OF THE I-TH PLY W.R.T. THE PLATE   STR00
C        (I.E., LAMINATE) COORDINATES.                                     STR00
C     R IS THE LOAD FACTOR (I.E.,THE STRENGTH RATIO) FOR THE GIVEN         STR00
C        TEMPERATURE DISTRIBUTION AND UNSCALED INPUT LOADING. R GREATER    STR00
C        THAN 1. IS ASSOCIATED W. SAFE LOADING, AND R LESS THAN 1. MEANS   STR00
```

```
   FAILURE HAS OCCURRED. FAILURE OCCURS WHEN ABS(MIN(R(I))) IS
   EQUAL TO UNITY.
STRES1(I),STRES2(I),STRES3(I) ARE THE STRESSES IN THE I-TH PLY W.R.T
   THE LAMINATE AXES.
STRESX(I),STRESY(I), AND TAUXY(I) ARE THE STRESSES IN THE I-TH PLY
   W.R.T. THE NATURAL (PLY) AXES, AFTER SCALING WITH THE LOAD FACTOR.
STRSL FOLLOWED BY A LAMINA OR LAMINATE "SUBSCRIPT" (I.E. STRSLX OR
   STRSL1) IS THE PLY STRESS DUE TO LOAD W.R.T. LAMINA OR LAMINATE
   COORDINATES BEFORE SCALING WITH THE LOAD FACTOR.
STRST FOLLOWED BY A LAMINA OR LAMINATE "SUBSCRIPT" IS THE PLY STRESS
   DUE TO TEMPERATURE W.R.T. LAMINA OR LAMINATE COORDINATES
STFRTM IS THE STRESS FREE TEMPERATURE
TEM(I) IS THE TEMPERATURE OF THE I-TH PLY.
THICK IS THE PLATE THICKNESS
X,XPRIME,Y,YPRIME,SS, AND FXYSTR ARE THE FAILURE PARAMETERS DETER-
   MINED EXPERIMENTALLY.
Z(I) AND Z(I+1) DENOTE THE LOCATIONS OF THE BOTTOM AND TOP
   SURFACES OF THE I-TH PLY, RESPECTIVELY.
ZN1,ZN2, AND ZN3 ARE THE UNSCALED INPLANE LOADS
ZNUXY,ZNUYX,EX,EY, AND G ARE THE STIFFNESS COEFFICIENTS OF A PLY
   W.R.T. PLY AXES.

   READ(5,40) NPLY,ITYPE,ZNUXY,EX,EY,G,ALFAX,ALFAY,STFRTM
   ZNUYX=ZNUXY*EY/EX
   WRITE (6,50) NPLY,ITYPE
   WRITE (6,60)
   WRITE (6,70)
   WRITE (6,71)
   WRITE (6,80) (ZNUXY,ZNUYX, EX,EY,G)
   WRITE (6,90) ALFAX,ALFAY
   WRITE (6,100) STFRTM
   WRITE(6,105)
   NN=NPLY+1
   NPL=NPLY

IF ITYPE=0 ALL PLIES ARE THE SAME THICKNESS
IF ITYPE=1 PLIES HAVE DIFFERENT THICKNESSES

   ICYCLE=1
   Z(1)=0.
   IF (ITYPE.NE.0) GO TO 20
   READ (5,110) THICK
   PLT=THICK/NPLY
   DO 10 I=1,NPLY
   IT(I)=0
   PLYTHK(I)=PLT
   Z(I+1)=Z(I)+PLT
   CONTINUE
   CONTINUE
   IF (ITYPE.NE.1) GO TO 30
   READ (5,120) (J,PLYTHK(I),I=1,NPLY)
   DO 25 I=1,NPLY
   Z(I+1)=Z(I)+PLYTHK(I)
   IT(I)=0
   CONTINUE
```

```
30    READ (5,130) (J,ANGLE(I),TEM(I),I=1,NPLY)                       STR01
      WRITE (6,65)                                                    STR01
      WRITE (6,75) (J,ANGLE(J),PLYTHK(J),TEM(J),J=1,NPL)              STR01
      WRITE (6,85)                                                    STR01
      WRITE (6,95) (Z(I),I=1,NN)                                      STR01
C                                                                     STR01
C    INPUT THE APPLIED INPLANE LOADS                                  STR01
C                                                                     STR01
      READ (5,140) ZN1,ZN2,ZN3                                        STR01
      READ(5,140) ZM1,ZM2,ZM3                                         STR01
      WRITE(6,210)                                                    STR01
      WRITE(6,220) ZN1,ZN2,ZN3                                        STR01
      WRITE(6,230)                                                    STR01
      WRITE(6,240) ZM1,ZM2,ZM3                                        STR01
      WRITE(6,245)                                                    STR01
C                                                                     STR01
C    LAMINA STIFFNESS COEFFICIENTS IN NATURAL (X,Y) COORDINATES       STR01
C                                                                     STR01
      DENOM=1.0-ZNUXY*ZNUYX                                           STR01
      QXX=EX/DENOM                                                    STR01
      QXY=(ZNUXY*EY)/DENOM                                            STR01
      QYY=EY/DENOM                                                    STR01
      QSS=G                                                           STR01
C                                                                     STR01
      WRITE (6,150)                                                   STR01
      WRITE (6,160) QXX,QYY,QXY,QSS                                   STR01
C                                                                     STR01
C    CALCULATE THE COEFFICIENTS OF THE FAILURE CRITERION              STR01
C                                                                     STR01
      READ (5,170) X,XPRIME,Y,YPRIME,SS,FXYSTR                        STR01
      WRITE(6,175)                                                    STR01
      WRITE (6,180) X,XPRIME,Y,YPRIME,SS,FXYSTR                       STR01
C                                                                     STR01
      FXX=1./(X*XPRIME)                                               STR01
      FX=(1./X)-(1./XPRIME)                                           STR01
      FYY=1./(Y*YPRIME)                                               STR01
      FY=(1./Y)-(1./YPRIME)                                           STR01
      FXY=FXYSTR*SQRT(FXX*FYY)                                        STR01
      FSS=1./(SS**2)                                                  STR01
C                                                                     STR01
      WRITE (6,195)                                                   STR01
      WRITE (6,190)                                                   STR01
      WRITE (6,200) FXX,FX,FYY,FY,FSS,FXY                             STR01
C                                                                     STR01
      CALL CYCLE                                                      STR01
C                                                                     STR01
C                                                                     STR01
      RETURN                                                          STR01
C                                                                     STR01
C                                                                     STR01
40    FORMAT (2I3,7G10.4)                                             STR01
50    FORMAT (1H0,2X,28HNO. OF PLIES ABOVE MIDPLANE=,I3,10X,6HITYPE=,I3)STR01
60    FORMAT (1H0,2X,35HITYPE=0 FOR UNIFORM PLY THICKNESSES)          STR01
70    FORMAT (3X,39HITYPE=1 FOR NON UNIFORM PLY THICKNESSES)          STR01
71    FORMAT(//1H0,2X,'LAMINA ELASTIC PROPERTIES W.R.T. LAMINA COORDINATSTR01
```

```
   &ES:')
    FORMAT (1H0,2X,5HNUXY=,G9.3,2X,5HNUYX=,G9.3,2X,3HEX=,G9.4,3X,3HEY=
   1,G9.4,3X,2HG=,G9.4)
    FORMAT (1H0,2X,7HALPHAX=,G10.4,10X,7HALPHAY=,G10.4)
0   FORMAT (//1H0,2X,'THE STRESS FREE TEMPERATURE IS',G10.4)
5   FORMAT(//1H0,2X,'LAMINATE CONSTRUCTION:')
0   FORMAT (G10.4)
0   FORMAT (I3,G10.4)
0   FORMAT (I3,2G10.4)
0   FORMAT (3G10.4)
0   FORMAT (//1H0,2X,46HSTIFFNESS COEFFICIENTS IN NATURAL COORDINATES:
   1)
0   FORMAT (1H0,5X,4HQXX=,G10.3,2X,4HQYY=,G10.3,2X,4HQXY=,G10.3,2X,4HQ
   1SS=,G10.3)
0   FORMAT (6G10.4)
5   FORMAT(//1H0,2X,'STRENGTHS W.R.T. LAMINA AXES:')
0   FORMAT (1H0,4X,2HX=,G10.4,3X,7HXPRIME=,G10.4,3X,2HY=,G10.4,3X,7HYP
   1RIME=,G10.4,3X,3HSS=,G10.4,3X,7HFXYSTR=,G10.4)
0   FORMAT (1H0,7X,3HFXX,11X,2HFX,11X,3HFYY,10X,2HFY,10X,3HFSS,10X,3HF
   1XY)
5   FORMAT(//1H0,2X,'TSAI-WU STRENGTH PARAMETERS W.R.T. LAMINA AXES:')
0   FORMAT (4X,G10.4,3X,G10.4,3X,G10.4,3X,G10.4,3X,G10.4,3X,G10.4)
0   FORMAT(//1H0,2X,'INPLANE LOADS W.R.T. LAMINATE AXES:')
0   FORMAT(1H0,5X,'ZN1=',G10.4,5X,'ZN2=',G10.4,5X,'ZN3=',G10.4)
0   FORMAT(//1H0,2X,'MOMENTS W.R.T. LAMINATE AXES:')
0   FORMAT(1H0,5X,'ZM1=',G10.4,5X,'ZM2=',G10.4,5X,'ZM3=',G10.4)
5   FORMAT(1H0,2X,'*** MULTIPLICATION OF THE ABOVE LOADS AND MOMENTS B
   1Y THE STRENGTH RATIO AT LAMINATE FAILURE GIVES THE FAILURE LOADS
   2AND MOMENTS')
    FORMAT (1H0,9X,3HPLY,10X,5HANGLE,11X,13HPLY THICKNESS,7X,3HTEM)
    FORMAT (9X,I3,10X,G10.4,9X,G10.4,6X,G10.4)
    FORMAT (1H0,2X,10HZ-LOCATION)
    FORMAT (5X,G10.4)
    END


    SUBROUTINE CYCLE

  SUBROUTINE CYCLE CONSTRUCTS THE A AND D MATRICES, AND THEN SOLVES
  FOR THE STRESSES IN EACH PLY. AFTER THE STRESSES ARE CALCULATED,
  SUBROUTINE FAILUR IS CALLED TO DETERMINE WHICH, IF ANY, PLIES
  HAVE FAILED. SUBROUTINE FAILUR IN TURN CALLS SUBROUTINE CYCLE
  UNTIL ALL PLIES WHICH WILL FAIL DUE TO THE LOAD AND THERMAL
  ENVIRONMENT HAVE FAILED.

    COMMON ANGLE(100),Z(101),A(3,3),D(3,3),AT(3,3),DT(3,3),AINV(3,3),W
   1K(50),DINV(3,3),PLYTHK(100),QBAR11(100),QBAR12(100),QBAR22(100),QB
   2AR16(100),QBAR26(100),QBAR66(100),ALPHA1(100),ALPHA2(100),ALPHA3(1
   300),DELTEM(100),STRES1(100),STRES2(100),STRES3(100),TEM(100),STRES
   4X(100),STRESY(100),TAUXY(100),R1(100),R2(100),ZN1,ZN2,ZN3,FXX,FX,F
   5YY,FY,FSS,FXY,QXX,QXY,QYY,QSS,ALFAX,ALFAY,STFRTM,STRSL1(100),
   6STRSL2(100),STRSL3(100),STRST1(100),STRST2(100),STRST3(100),
   7STRSTX(100),STRSTY(100),TAUTXY(100),STRSLX(100),STRSLY(100),TAULXY
   8(100),R(70),RMAX,ZM1,ZM2,ZM3,
   9NPLY,ICYCLE,IFAIL,NPL,IT(100)
```

```
C
      WRITE (6,50) ICYCLE
      NN=NPL+1
C
C   ZERO OUT THE A AND D MATRICES, AND THE THERMAL LOAD VECTOR
C
      A(1,1)=0.
      A(1,2)=0.
      A(1,3)=0.
      A(2,2)=0.
      A(2,3)=0.
      A(3,3)=0.
      D(1,1)=0.
      D(1,2)=0.
      D(1,3)=0.
      D(2,2)=0.
      D(2,3)=0.
      D(3,3)=0.
      ZNT1=0.
      ZNT2=0.
      ZNT3=0.
C
C   LAMINA STIFFNESS COEFFICIENTS W.R.T. LAMINATE COORDINATES
C
      WRITE (6,100)
      WRITE (6,110)
C
      DO 20 IPLY=1,NPLY
      IF(IT(IPLY).EQ.1) GO TO 20
      DELTEM(IPLY)=TEM(IPLY)-STFRTM
      ANGLE1=ANGLE(IPLY)/57.29578
      S=SIN(ANGLE1)
      S2=S**2
      S3=S*S2
      S4=S2**2
      C=COS(ANGLE1)
      C2=C**2
      C3=C2*C
      C4=C2**2
      CSSQ=C2*S2
      QBAR11(IPLY)=QXX*C4+2.*(QXY+2.*QSS)*CSSQ+QYY*S4
      QBAR12(IPLY)=(QXX+QYY-4.*QSS)*CSSQ+QXY*(S4+C4)
      QBAR22(IPLY)=QXX*S4+2.*(QXY+2.*QSS)*CSSQ+QYY*C4
      QBAR16(IPLY)=(QXX-QXY-2.*QSS)*S*C3+(QXY-QYY+2.*QSS)*S3*C
      QBAR26(IPLY)=(QXX-QXY-2.*QSS)*S3*C+(QXY-QYY+2.*QSS)*S*C3
      QBAR66(IPLY)=(QXX+QYY-2.*QXY-2.*QSS)*CSSQ+QSS*(S4+C4)
C
      ALPHA1(IPLY)=C2*ALFAX+S2*ALFAY
      ALPHA2(IPLY)=S2*ALFAX+C2*ALFAY
      ALPHA3(IPLY)=(ALFAY-ALFAX)*C*S
C
      WRITE (6,120) IPLY,QBAR11(IPLY),QBAR22(IPLY),QBAR12(IPLY),QBAR66(I
     1PLY),QBAR16(IPLY),QBAR26(IPLY)
C
C   CONSTRUCT THE THERMAL LOAD VECTOR, ZNT
```

```
      P=PLYTHK(IPLY)
      DTM=DELTEM(IPLY)
      A1=ALPHA1(IPLY)
      A2=ALPHA2(IPLY)
      A3=ALPHA3(IPLY)
      ZNT1=ZNT1+P*DTM*(QBAR11(IPLY)*A1+QBAR12(IPLY)*A2+QBAR16(IPLY)*A3)
      ZNT2=ZNT2+P*DTM*(QBAR12(IPLY)*A1+QBAR22(IPLY)*A2+QBAR26(IPLY)*A3)
      ZNT3=ZNT3+P*DTM*(QBAR16(IPLY)*A1+QBAR26(IPLY)*A2+QBAR66(IPLY)*A3)

 CONSTRUCTION OF LAMINATE A AND D MATRICES FROM LAMINA MATRICES

      Z1=Z(IPLY+1)
      Z2=Z(IPLY)
      THICK=2.*(Z1-Z2)
      RK=2.*((Z1**3)-(Z2**3))/3.

      A(1,1)=A(1,1)+THICK*QBAR11(IPLY)
      A(1,2)=A(1,2)+THICK*QBAR12(IPLY)
      A(1,3)=A(1,3)+THICK*QBAR16(IPLY)
      A(2,1)=A(1,2)
      A(2,2)=A(2,2)+THICK*QBAR22(IPLY)
      A(2,3)=A(2,3)+THICK*QBAR26(IPLY)
      A(3,1)=A(1,3)
      A(3,2)=A(2,3)
      A(3,3)=A(3,3)+THICK*QBAR66(IPLY)

      D(1,1)=D(1,1)+RK*QBAR11(IPLY)
      D(1,2)=D(1,2)+RK*QBAR12(IPLY)
      D(1,3)=D(1,3)+RK*QBAR16(IPLY)
      D(2,1)=D(1,2)
      D(2,2)=D(2,2)+RK*QBAR22(IPLY)
      D(2,3)=D(2,3)+RK*QBAR26(IPLY)
      D(3,1)=D(1,3)
      D(3,2)=D(2,3)
      D(3,3)=D(3,3)+RK*QBAR66(IPLY)

      CONTINUE

      ZNT1=2.*ZNT1
      ZNT2=2.*ZNT2
      ZNT3=2.*ZNT3

 CALCULATE THE TOTAL INPLANE FORCES. TOTN1=ZN1+ZNT1

      TOTN1=ZN1+ZNT1
      TOTN2=ZN2+ZNT2
      TOTN3=ZN3+ZNT3

      WRITE(6,125)
      WRITE (6,130) ZN1,ZNT1,TOTN1
      WRITE (6,140) ZN2,ZNT2,TOTN2
      WRITE (6,150) ZN3,ZNT3,TOTN3

      WRITE (6,155)
```

```
      WRITE (6,160)
      DO 25 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 25
      WRITE (6,170)  I,ALPHA1(I),ALPHA2(I),ALPHA3(I)
25    CONTINUE
C
      WRITE (6,180)
      WRITE (6,185) A(1,1),A(1,2),A(1,3),D(1,1),D(1,2),D(1,3)
      WRITE (6,190) A(2,1),A(2,2),A(2,3),D(2,1),D(2,2),D(2,3)
      WRITE (6,190) A(3,1),A(3,2),A(3,3),D(3,1),D(3,2),D(3,3)
C
C   THE A AND C MATRICES PER UNIT THICKNESS OF LAMINATE
C
      H=2.*Z(NN)
      WRITE (6,200) H
      AT(1,1)=A(1,1)/H
      AT(1,2)=A(1,2)/H
      AT(1,3)=A(1,3)/H
      AT(2,1)=AT(1,2)
      AT(2,2)=A(2,2)/H
      AT(2,3)=A(2,3)/H
      AT(3,1)=AT(1,3)
      AT(3,2)=AT(2,3)
      AT(3,3)=A(3,3)/H
C
      DT(1,1)=D(1,1)/H
      DT(1,2)=C(1,2)/H
      CT(1,3)=D(1,3)/H
      DT(2,1)=DT(1,2)
      CT(2,2)=D(2,2)/H
      CT(2,3)=D(2,3)/H
      CT(3,1)=DT(1,3)
      DT(3,2)=DT(2,3)
      DT(3,3)=C(3,3)/H
C     WRITE (6,210)
C     WRITE (6,185) AT(1,1),AT(1,2),AT(1,3),DT(1,1),DT(1,2),DT(1,3)
C     WRITE (6,190) AT(2,1),AT(2,2),AT(2,3),DT(2,1),DT(2,2),DT(2,3)
C     WRITE (6,190) AT(3,1),AT(3,2),AT(3,3),DT(3,1),DT(3,2),DT(3,3)
C
C   INVERSION CF THE A AND D MATRICES
C
      CALL LINV2F (A,3,3,AINV,5,WK,IER)
      CALL LINV2F (D,3,3,DINV,5,WK,IER)
C
      WRITE (6,220)
      WRITE (6,185) AINV(1,1),AINV(1,2),AINV(1,3),DINV(1,1),DINV(1,2),DI
     1NV(1,3)
      WRITE (6,190) AINV(2,1),AINV(2,2),AINV(2,3),DINV(2,1),DINV(2,2),DI
     1NV(2,3)
      WRITE (6,190) AINV(3,1),AINV(3,2),AINV(3,3),DINV(3,1),DINV(3,2),DI
     1NV(3,3)
C
C   CONSTRUCT THE EXPRESSION FOR PLY STRESSES DUE TO TEMPERATURE
C
C     1. AINV*ZNT
```

```
   WRITE(6,225)
   WRITE(6,230)
   DO 30 I=1,NPLY
   IF(IT(I).EQ.1) GO TO 30
   AINT1=AINV(1,1)*ZNT1+AINV(1,2)*ZNT2+AINV(1,3)*ZNT3
   AINT2=AINV(2,1)*ZNT1+AINV(2,2)*ZNT2+AINV(2,3)*ZNT3
   AINT3=AINV(3,1)*ZNT1+AINV(3,2)*ZNT2+AINV(3,3)*ZNT3

   2. ALPHA*DELTEM

   ALTM1=ALPHA1(I)*DELTEM(I)
   ALTM2=ALPHA2(I)*DELTEM(I)
   ALTM3=ALPHA3(I)*DELTEM(I)

   3.  THERMAL STRESS= QBAR*(AINV*ZNT-ALPHA*DELTEM)

   A1=AINT1-ALTM1
   A2=AINT2-ALTM2
   A3=AINT3-ALTM3
   STRST1(I)=QBAR11(I)*A1+QBAR12(I)*A2+QBAR16(I)*A3
   STRST2(I)=QBAR12(I)*A1+QBAR22(I)*A2+QBAR26(I)*A3
   STRST3(I)=QBAR16(I)*A1+QBAR26(I)*A2+QBAR66(I)*A3
   WRITE (6,240)  I,STRST1(I),STRST2(I),STRST3(I),DELTEM(I)
   CONTINUE

 CALCULATE THE PLY STRESSES DUE TO LOADS; STRSL=Q*AINV*ZN

   DO 35 I=1,NPLY
   IF(IT(I).EQ.1) GO TO 35

   AINL1=AINV(1,1)*ZN1+AINV(1,2)*ZN2+AINV(1,3)*ZN3
   AINL2=AINV(2,1)*ZN1+AINV(2,2)*ZN2+AINV(2,3)*ZN3
   AINL3=AINV(3,1)*ZN1+AINV(3,2)*ZN2+AINV(3,3)*ZN3

   STRSL1(I)=QBAR11(I)*AINL1+QBAR12(I)*AINL2+QBAR16(I)*AINL3
   STRSL2(I)=QBAR12(I)*AINL1+QBAR22(I)*AINL2+QBAR26(I)*AINL3
   STRSL3(I)=QBAR16(I)*AINL1+QBAR26(I)*AINL2+QBAR66(I)*AINL3

   CONTINUE


 CALCULATE THE STRESSES DUE TO MOMENTS

   IF(ZM1.EQ.0..AND.ZM2.EQ.0..AND.ZM3.EQ.0.) GO TO 37
   DO 36 I=1,NPLY
   IF(IT(I).EQ.1) GO TO 36
   ZZ=(Z(I+1)+Z(I))/2.
   ZKAP1=DINV(1,1)*ZM1+DINV(1,2)*ZM2+DINV(1,3)*ZM3
   ZKAP2=DINV(2,1)*ZM1+DINV(2,2)*ZM2+DINV(2,3)*ZM3
   ZKAP3=DINV(3,1)*ZM1+DINV(3,2)*ZM2+DINV(3,3)*ZM3

   DINM1=ZZ*ZKAP1
   DINM2=ZZ*ZKAP2
   DINM3=ZZ*ZKAP3
```

```
C
C    ADD TC STRESSES DUE TO INPLANE LOADS
C
C
      STRSL1(I)=STRSL1(I)+QBAR11(I)*DINM1+QBAR12(I)*DINM2+QBAR16(I)*DINM
     &3
      STRSL2(I)=STRSL2(I)+QBAR12(I)*DINM1+QBAR22(I)*DINM2+QBAR26(I)*DINM
     &3
      STRSL3(I)=STRSL3(I)+QBAR16(I)*DINM1+QBAR26(I)*DINM2+QBAR66(I)*DINM
     &3

36    CONTINUE
37    CONTINUE
      WRITE(6,270)
      DO 38 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 38
      WRITE(6,280)  I,STRSL1(I),STRSL2(I),STRSL3(I)
38    CONTINUE
C
C
C    FAILURE ANALYSIS
C    9 1. CONVERT PLY STRESSES TO NATURAL (LAMINA) AXES
      DO 40 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 40
C
      ANGLE1=ANGLE(I)/57.29578
      S=SIN(ANGLE1)
      S2=S**2
      C=COS(ANGLE1)
      C2=C**2
      SC=S*C
      STRSTX(I)=STRST1(I)*C2+STRST2(I)*S2+STRST3(I)*(2.*SC)
      STRSTY(I)=STRST1(I)*S2+STRST2(I)*C2-STRST3(I)*(2.*SC)
      TAUTXY(I)=SC*(STRST2(I)-STRST1(I))+STRST3(I)*(C2-S2)
C
      STRSLX(I)=STRSL1(I)*C2+STRSL2(I)*S2+STRSL3(I)*(2.*SC)
      STRSLY(I)=STRSL1(I)*S2+STRSL2(I)*C2-STRSL3(I)*(2.*SC)
      TAULXY(I)=SC*(STRSL2(I)-STRSL1(I))+STRSL3(I)*(C2-S2)
C
C     2. CALCULATE THE COEFFICIENTS OF THE QUADRATIC:A*(R(I)**2)
C          +B*R(I)+C=0.
C
      AA=FXX*(STRSLX(I)**2)+2.*FXY*(STRSLX(I)*STRSLY(I))+FYY*(STRSLY(I)*
     1*2)+FSS*(TAULXY(I)**2)
      B=2.*(FXX*STRSLX(I)*STRSTX(I)+FXY*STRSLX(I)*STRSTY(I)+
     1FXY*STRSLY(I)*STRSTX(I)+FYY*STRSLY(I)*STRSTY(I)+FSS*TAULXY(I)
     2*TAUTXY(I))+FX*STRSLX(I)+FY*STRSLY(I)
      C=FXX*(STRSTX(I)**2)+2.*FXY*STRSTX(I)*STRSTY(I)+FYY*(STRSTY(I)**2)
     1+FSS*(TAUTXY(I)**2)+FX*STRSTX(I)+FY*STRSTY(I)-1.
C
C     3. SOLVE THE FAILURE QUADRATIC FOR THE STRENGTH RATIOS, R(I)
C
      CC=SQRT((B**2)-4.*AA*C)
      R1(I)=(-B+CC)/(2.*AA)
      R2(I)=(-B-CC)/(2.*AA)
C
```

```
      CONTINUE

      WRITE (6,250)
      DO 41 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 41
      WRITE (6,260)  I,STRSTX(I),STRSTY(I),TAUTXY(I),R1(I),R2(I)
      CONTINUE

      WRITE(6,290)
      DO 42 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 42
      WRITE(6,300)  I,STRSLX(I),STRSLY(I),TAULXY(I)
      CONTINUE

      CALL FAILUR

      RETURN

      FORMAT (1H1,1X,7HICYCLE=,I3)
      FORMAT (//1H0,2X,'PLY STIFFNESS COEFFICIENTS W.R.T. LAMINATE AXES:
     1')
      FORMAT (1H0,3X,3HPLY,3X,6HQBAR11,5X,6HQBAR22,5X,6HQBAR12,5X,6HQBAR
     166,5X,6HQBAR16,5X,6HQBAR26)
      FORMAT (2X,I3,6(1X,G10.3))
      FORMAT(//1H0,2X,'APPLIED, THERMAL, AND TOTAL INPLANE FORCES:')
      FORMAT (1H0,5X,4HZN1=,G10.4,10X,5HZNT1=,G10.4,10X,6HTOTN1=,G10.4)
      FORMAT (6X,4HZN2=,G10.4,10X,5HZNT2=,G10.4,10X,6HTOTN2=,G10.4)
      FORMAT (6X,4HZN3=,G10.4,10X,5HZNT3=,G10.4,10X,6HTOTN3=,G10.4)
      FORMAT (//1H0,2X,'THERMAL COEFFICIENTS OF EXPANSION W.R.T. LAMINAT
     1E AXES:')
      FORMAT (1H0,3X,3HPLY,11X,6HALPHA1,14X,6HALPHA2,14X,6HALPHA6)
      FORMAT (3X,I3,10X,G10.4,10X,G10.4,10X,G10.4)
      FORMAT (//1H0,12X,8HA-MATRIX,35X,8HD-MATRIX)
      FORMAT (1H0,1X,G11.4,1X,G11.4,1X,G11.4,8X,G11.4,1X,G11.4,1X,G11.4)
      FORMAT (2X,G11.4,1X,G11.4,1X,G11.4,8X,G11.4,1X,G11.4,1X,G11.4)
      FORMAT (//1H0,5X,16HPLATE THICKNESS=,G10.4)
      FORMAT (/1H0,9X,18HA-MATRIX/THICKNESS,25X,18HD-MATRIX/THICKNESS)
      FORMAT (1H0,9X,16HA-INVERSE MATRIX,30X,16HD-INVERSE MATRIX)
      FORMAT(//1H0,2X,'STRESSES IN LAMINA AND LAMINATE COORDINATES BEFOR
     &E MULTIPLICATION WITH THE LOAD FACTOR:')
      FORMAT (1H0,3X,3HPLY,6X,6HSTRST1,12X,6HSTRST2,14X,6HSTRST3,9X,6HDE
     1LTEM)
      FORMAT (4X,I3,7X,G10.4,8X,G10.4,9X,G10.4,5X,G10.4)
      FORMAT (1H0,3X,3HPLY,6X,6HSTRSTX,9X,6HSTRSTY,11X,6HTAUTXY,12X,2HR1
     1,12X,2HR2)
      FORMAT (4X,I3,5X,G10.4,5X,G10.4,5X,G10.4,5X,G10.4,5X,G10.4)
      FORMAT( 1H0,3X,3HPLY,6X,6HSTRSL1,12X,6HSTRSL2,14X,6HSTRSL3)
      FORMAT(4X,I3,7X,G10.4,8X,G10.4,9X,G10.4)
      FORMAT( 1H0,3X,3HPLY,6X,6HSTRSLX,9X,6HSTRSLY,11X,6HTAULXY)
      FORMAT( 4X,I3,5X,G10.4,5X,G10.4,5X,G10.4)
      END

      SUBROUTINE FAILUR
```

```
C     THIS SUBROUTINE DETERMINES WHICH PLIES HAVE FAILED BY USING
C     THE TSAI-WU FAILURE CRITERION. THE PLY WHICH HAS FAILED IS
C     ELIMINATED AND THE ANALYSIS IS RETURNED TO SUBROUTINE CYCLE,
C     WHICH RECALCULATES THE STRESSES FOR THE MODIFIED LAMINATE AND
C     RETURNS TO SUBROUTINE FAILUR AGAIN. THIS PROCESS IS REPEATED
C     UNTIL ALL PLIES WHICH ARE TO FAIL HAVE FAILED.
C
C
C     THIS SUBROUTINE CHECKS FOR PLY FAILURE.
C
      COMMON ANGLE(100),Z(101),A(3,3),D(3,3),AT(3,3),DT(3,3),AINV(3,3),W
     1K(50),DINV(3,3),PLYTHK(100),QBAR11(100),QBAR12(100),QBAR22(100),QB
     2AR16(100),QBAR26(100),QBAR66(100),ALPHA1(100),ALPHA2(100),ALPHA3(1
     300),DELTEM(100),STRES1(100),STRES2(100),STRES3(100),TEM(100),STRES
     4X(100),STRESY(100),TAUXY(100),R1(100),R2(100),ZN1,ZN2,ZN3,FXX,FX,F
     5YY,FY,FSS,FXY,QXX,QXY,QYY,QSS,ALFAX,ALFAY,STFRTM,STRSL1(100),
     6STRSL2(100),STRSL3(100),STRST1(100),STRST2(100),STRST3(100),
     7STRSTX(100),STRSTY(100),TAUTXY(100),STRSLX(100),STRSLY(100),TAULXY
     8(100),R(70),RMAX,ZM1,ZM2,ZM3,
     9NPLY,ICYCLE,IFAIL,NPL,IT(100)
C
      R(ICYCLE)=1.E+20
C
      DO 10 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 10
      ABSR1=ABS(R1(I))
      ABSR2=ABS(R2(I))
      ABSR=AMIN1(ABSR1,ABSR2)
      IF(ABSR.LT.R(ICYCLE)) IFAIL=I
      IF(ABSR.LT.R(ICYCLE)) R(ICYCLE)=ABSR
10    CONTINUE
C
      WRITE(6,110) ICYCLE,IFAIL,R(ICYCLE)
C
C     CALCULATE THE STRESSES AT IMPENDING PLY FAILURE
C
      DO 20 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 20
      STRESX(I)=R(ICYCLE)*STRSLX(I)+STRSTX(I)
      STRESY(I)=R(ICYCLE)*STRSLY(I)+STRSTY(I)
      TAUXY(I)=R(ICYCLE)*TAULXY(I)+TAUTXY(I)
20    CONTINUE
C
      WRITE(6,100)
      WRITE(6,120)
      DO 30 I=1,NPLY
      IF(IT(I).EQ.1) GO TO 30
      WRITE(6,125) (I,STRESX(I),STRESY(I),TAUXY(I))
30    CONTINUE
C
C     CALCULATE THE STRENGTH RATIO FOR LAMINATE FAILURE
C
      IF(ICYCLE.EQ.1) RMAX=R(1)
      RMAX=AMAX1(R(ICYCLE),RMAX)
C
```

```
  CHECK FOR LAMINATE FAILURE

     NPL=NPL-1
     IF(NPL.NE.0) GO TO 50
     WRITE(6,130) RMAX
     IF(NPL.EQ.0) STOP 1
     ICYCLE =ICYCLE+1
     IT(IFAIL)=1

     CALL CYCLE

     RETURN

0    FORMAT(//1H0,2X,'STRESSES AT IMPENDING PLY FAILURE AFTER MULTIPLIC
    &ATION W. THE LOAD FACTOR:')
0    FORMAT(//1H0,2X,'CYCLE=',I3,5X,'IFAIL=',I3,5X,'LOAD FACTOR=',G10.4
    1)
0    FORMAT( 1H0,3X,'PLY',8X,'STRESX',8X,' STRESY',10X,'TAUXY')
5    FORMAT(4X,I3,6X,G10.4,4X,G10.4,4X,G10.4)
0    FORMAT(//////1H0,2X,'***  THE STRENGTH RATIO FOR THE LAMINATE IS'
    &,G10.4)

     END
```

# Appendix C

## Instructions for Use of the Computer Program

The following describes the sequence of input data for the computer program:

1. Read (5,40) NPLY, ITYPE, ZNUXY, EX, EY, ALFAX, ALFAY, STFRTM

   40 Format (2I3, 7G10.4)

   NPLY = Number of plies above the laminate midplane
   ITYPE is 0 if all plies are of equal thickness;
   ITYPE is 1 if plies have different thicknesses
   ZNUXY ($\nu_{xy}$) = lamina longitudinal Poissons ratio
   EX ($E_x$) = lamina longitudinal stiffness
   EY ($E_y$) = lamina transverse stiffness
   ALFAX ($\alpha_x$) = lamina longitudinal coefficient of thermal expansion
   ALFAY ($\alpha_y$) = lamina transverse coefficient of thermal expansion
   STFRTM = stress free temperature (i.e., the curing temperature of the laminate)

2. Read (5,110) THICK (not inputted if ITYPE = 1)

   110 Format (G10.4)

   THICK = laminate thickness

3. Read (5,120) (J,PLYTHK (I), I = 1, NPLY) (not inputted if ITYPE = 0)

   120 FORMAT (I3, G10.4)

   J = ply number
   PLYTHK (I) = ply thickness

4. Read (5,130) (J, ANGLE (I), TEM (I), I = 1, NPLY)

   130 Format (I3, 2G10.4)

   J = ply number
   ANGLE (I) = $\phi_i$ = ply orientation angle
   TEM (I) = $T_i$ = ply temperature

5. Read (5,140) ZN1, ZN2, ZN3

   140 Format (3610.4)

   ZN1 = $N_1$ = normal force along laminate 1-axis

   ZN2 = $N_2$ = normal force along laminate 2-axis

   ZN3 = $N_3$ = shear force on laminate

6. Read (6,140) ZM1, ZM2, ZM3

   ZM1 = $M_1$ = bending moment on laminate 1-face

   ZM2 = $M_2$ = bending moment on laminate 2-face

   ZM3 = $M_3$ = twisting moment on laminate

7. Read (5,170) X, XPRIME, Y, YPRIME, SS, FXYSTR

   170 Format (6G10.4)

   X = longitudinal tensile strength of lamina

   XPRIME (X') = longitudinal compressive strength of lamina

   Y = transverse tensile strength of lamina

   YPRIME = (Y') = transverse compressive strength of lamina

   SS (S) = lamina shear strength

   FXYSTR ($F^*_{xy}$) = 0.S (in accordance with Eq. (59))

A sample output of a typical analysis is presented on the pages following this page. This is the problem of a cross-ply E-glass/epoxy laminate subjected to uniaxial loading, i.e. $(N_1, N_2, N_3) = (1,0,0)$. The cross-ply ratio, M, is 0.2. The curing temperature is 270$^{\circ}$F, and the temperature of the laminate is 70$^{\circ}$F.

```
NO. CF PLIES ABOVE MIDPLANE=  2          ITYPE=  1
ITYPE=0 FCR UNIFORM PLY THICKNESSES
ITYPE=1 FCR NON UNIFCRM PLY THICKNESSES

LAMINA ELASTIC PROPERTIES W.R.T. LAMINA COORDINATES:
NUXY= .250      NUYX= .833D-01  EX=.7800D+07   EY=.2600D+07   G=.1250D+07
ALPHAX= .3500D-05            ALPHAY= .1140D-04

THE STRESS FREE TEMPERATURE IS 270.0

LAMINATE CONSTRUCTION:
      PLY           ANGLE            PLY THICKNESS       TEM
       1            90.00              41.67            70.00
       2            .0                 .8333D-01        70.00
Z-LOCATICN
   .0
  41.67
  41.75

INPLANE LOADS W.R.T. LAMINATE AXES:
   ZN1= 1.000          ZN2= .0              ZN3= .0

MOMENTS W.R.T. LAMINATE AXES:
   ZM1= .0             ZM2= .0              ZM3= .0
*** MULTIPLICATION OF THE ABOVE LOADS AND MOMENTS BY THE STRENGTH RATIO AT LAMINATE FAILURE GIVES THE FAILURE LOADS  AND MOMENTS

STIFFNESS COEFFICIENTS IN NATURAL COORDINATES:
   QXX=  .797D+07  QYY=  .266D+07  QXY=  .664D+06  QSS=  .125D+07

STRENGTHS W.R.T. LAMINA AXES:
  X= .1500D+06   XPRIME= .1500D+06   Y= 4000.        YPRIME= .2000D+05   SS= 6000.      FXYSTR=-.5000

TSAI-WU STRENGTH PARAMETERS W.R.T. LAMINA AXES:
     FXX            FX          FYY          FY           FSS          FXY
  .4444D-10    .0           .1250D-07    .2000D-03    .2778D-07   -.3727D-09
```

```
ICYCLE=  1

PLY STIFFNESS COEFFICIENTS W.R.T. LAMINATE AXES:

PLY    QBAR11      QBAR22      QBAR12      QBAR66      QBAR16      QBAR26
 1    .266D+07    .797D+07    .664D+06    .125D+07    .679D-02    .641D-01
 2    .797D+07    .266D+07    .664D+06    .125D+07    .0          .0

APPLIED, THERMAL, AND TOTAL INPLANE FORCES:

   ZN1= 1.000                ZNT1=-.5444D+06            TOTN1=-.5444D+06
   ZN2= .0                   ZNT2=-.5919D+06            TOTN2=-.5919D+06
   ZN3= .0                   ZNT3=-.7226D-02            TOTN3=-.7226D-02

THERMAL COEFFICIENTS OF EXPANSION W.R.T. LAMINATE AXES:

PLY            ALPHA1             ALPHA2             ALPHA6
 1            .1140D-04          .3500D-05          .1055D-12
 2            .3500D-05          .1140D-04          .0

          A-MATRIX                                        D-MATRIX

.2226D+09    .5543D+08    .5657                 .1304D+12    .3221D+11    327.4
.5543D+08    .6643D+09    5.342                 .3221D+11    .3849D+12    3092.
.5657        5.342        .1044D+09             327.4        3092.        .6065D+11

   PLATE THICKNESS= 83.50

      A-INVERSE MATRIX                               D-INVERSE MATRIX

 .4588D-08   -.3828D-09   -.5271D-17             .7833D-11   -.6553D-12   -.8874D-20
-.3828D-09    .1537D-08   -.7661D-16            -.6553D-12    .2653D-11   -.1317D-18
-.5271D-17   -.7661D-16    .9581D-08            -.8874D-20   -.1317D-18    .1649D-10

STRESSES IN LAMINA AND LAMINATE COORDINATES BEFORE MULTIPLICATION WITH THE LOAD FACTOR:

PLY       STRST1            STRST2              STRST3          DELTEM
 1         22.93            -6.297              .5255D-07       -200.0
 2        -.1147D+05         3148.             -.2627D-04       -200.0

PLY       STRSL1            STRSL2              STRSL3
 1         .1193D-01        -.4058D-05          .1318D-13
 2         .3629D-01         .2029D-02         -.6589D-11

PLY       STRSTX          STRSTY          TAUTXY            R1             R2
 1       -6.297           22.93          -.4427D-06       .3334D+06     -.1679D+07
 2       -.1147D+05       3148.          -.2627D-04       .4406D+06     -.8800D+07

PLY       STRSLX          STRSLY          TAULXY
 1       -.4058D-05       .1193D-01      -.1593D-09
 2        .3629D-01       .2029D-02      -.6589D-11

CYCLE=  1     IFAIL=  1     LOAD FACTOR= .3334D+06

STRESSES AT IMPENDING PLY FAILURE AFTER MULTIPLICATION W. THE LOAD FACTOR:

PLY       STRESX          STRESY          TAUXY
 1       -7.650           4000.          -.5355D-04
 2        633.9           3825.          -.2847D-04
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ICYCLE=  2

PLY STIFFNESS COEFFICIENTS W.R.T. LAMINATE AXES:
 PLY    QBAR11      QBAR22      QBAR12      QBAR66      QBAR16      QBAR26
 2    .797D+07    .266D+07    .664D+06    .125D+07    .0          .0

APPLIED, THERMAL, AND TOTAL INPLANE FORCES:
   ZN1= 1.000                 ZNT1=-1182.               TOTN1=-1181.
   ZN2= .0                    ZNT2=-1086.               TOTN2=-1086.
   ZN3= .0                    ZNT3= .0                  TOTN3= .0

THERMAL COEFFICIENTS OF EXPANSION W.R.T. LAMINATE AXES:
 PLY             ALPHA1                ALPHA2                ALPHA6
  2             .3500D-05             .1140D-04             .0

          A-MATRIX                                      D-MATRIX
 .1328D+07   .1106D+06   .0                 .2310D+10   .1925D+09   .0
 .1106D+06   .4425D+06   .0                 .1925D+09   .7698D+09   .0
 .0          .0          .2083D+06          .0          .0          .3624D+09

   PLATE THICKNESS= 83.33
      A-INVERSE MATRIX                             D-INVERSE MATRIX
 .7653D-06  -.1923D-06   .0                 .4422D-09  -.1106D-09   .0
-.1923D-06   .2308D-05   .0                -.1106D-09   .1327D-08   .0
 .0          .0          .4800D-05          .0          .0          .2759D-08

STRESSES IN LAMINA AND LAMINATE COORDINATES BEFORE MULTIPLICATION WITH THE LOAD FACTC
 PLY       STRST1           STRST2             STRST3          DELTEM
  2        -.2889D-09       -.2658D-09         .0              -200.0
 PLY       STRSL1           STRSL2             STRSL3
  2        6.000            .0                 .0
 PLY       STRSTX        STRSTY          TAUTXY          R1              R2
  2       -.2889D-09    -.2658D-09      .0             .2500D+05      -.2500D+05
 PLY       STRSLX        STRSLY          TAULXY
  2        6.000         .0             .0

CYCLE=  2     IFAIL=  2     LOAD FACTOR= .2500D+05

STRESSES AT IMPENDING PLY FAILURE AFTER MULTIPLICATION W. THE LOAD FACTOR:
 PLY          STRESX       STRESY         TAUXY
  2          .1500D+06    -.2658D-09     .0

***  THE STRENGTH RATIO FOR THE LAMINATE IS .3334D+06
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